TL1

Presentation to CAMP, March 2021

Slide 1

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My recent areas of physics research

- Applying coset space methods to spacetime symmetries
  - Compactification

    "Fully covariant spontaneous compactification" and RG posts

  - GR and teleparallelism

    "Tangent space symmetries in general relativity and teleparallelism"

- Roots of quantisation

    "Correspondence between Classical Field Theory in a finite universe and Quantum Mechanics – position, wavenumber and momentum"

Classical Field Theory $\longleftrightarrow$ QM (in finite universe) – position, wavenumber, momentum

- Format is a set of notes, rather than a paper
- Much of it is standard theory for Classical Field Theory, but
  - Brought together from different sources
  - Applied to a finite universe (with 1 spatial dimension for simplicity)

  $\Rightarrow$ appearance of quantum-like features slightly more cleanly than in standard theory, strong hints at dynamical interpretation of $\hbar$

- Questions:
  - Is there anything new here?
  - Are there flaws/holes in this analysis?
  - Is there anything that could be publishable in this?

Basic idea

- Eigenstates, superpositions and uncertainty relations usually seen as distinguishing features of QM
- Some authors see appearance of $\hbar$ as distinguishing feature, some just see $\hbar$ as scaling factor

BUT in classical field theory:

- Fourier analysis provides description of scalar field configuration as superposition of eigenstates of a derivative operator
- Uncertainty relations between position x and wavenumber k
- Can in theory define momentum density for a field configuration

Basic idea (cont'd) – and sneak peek at results

THEREFORE:

- Try calculating momentum density and integrating over finite universe to get finite value of momentum p
- See if this gives us scaling factor between p and k $\Rightarrow$ dynamical interpretation of $\hbar$

- For simplicity, we do this with 1 spatial dimension, in non-relativistic situation
- Very nearly works – unclear whether it would work completely in relativistic case

Action and Euler-Lagrange equations

- Start with actional functional. Main one we consider is

$$S = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \frac{1}{2}\rho \left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{1}{2}\rho v^2 \left(\frac{\partial \phi}{\partial x}\right)^2 \mathrm{d}x \, \mathrm{dt} \qquad (1)$$

- Leads to wave equation

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \phi}{\partial t^2} = 0$$

Solutions

- Wide class of solutions – any function of form

$$\phi = f_1(x - vt) + f_2(x + vt)$$

- Left-moving and right-moving parts

- Let $x = 0$ be centre of universe of radius R. Consider solutions localised around $x = 0$ at time $t = t_0$.
- Can find spectral decomposition at $t = t_0$ by taking ϕ to be part of waveform Φ with period $2R' \geq 2R$

Spectral decomposition

Fourier series

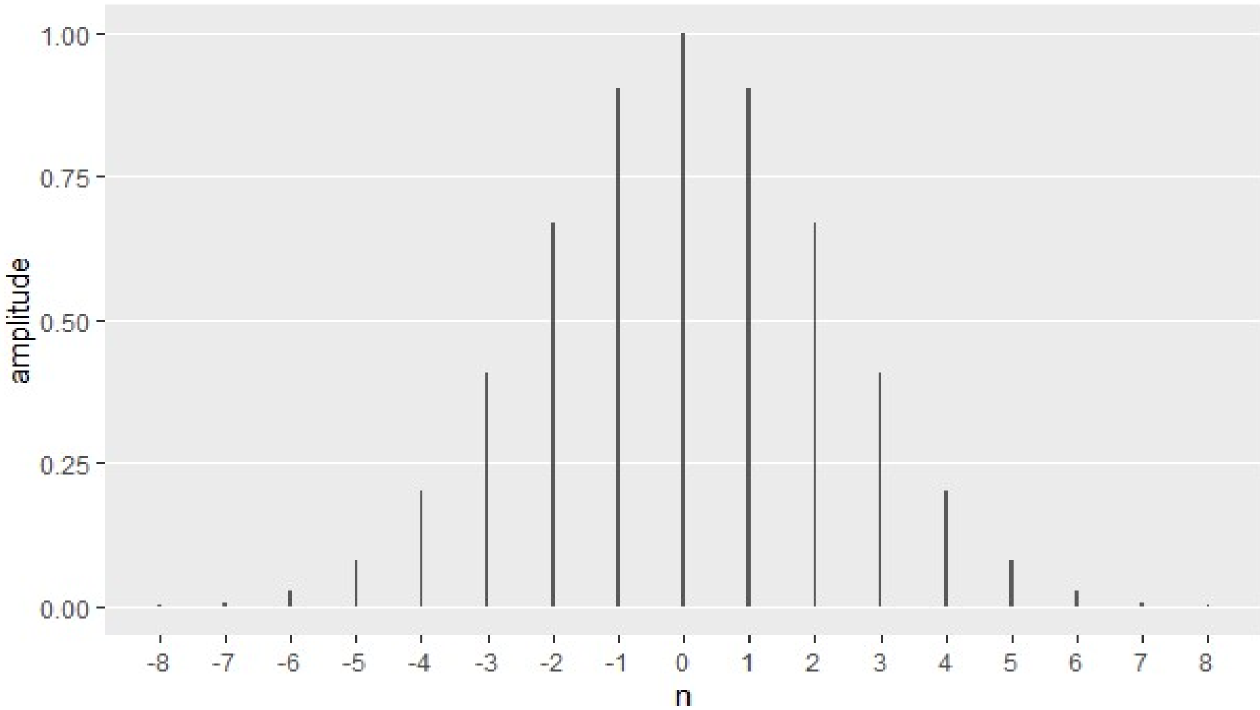
$$\Phi = \sum_{n=-\infty}^{\infty} c_n \; \mathrm{e}^{in\pi x/R'} \qquad (2)$$

Where

$$c_n = \frac{1}{2R'} \int_{-R'}^{R'} \Phi \; \mathrm{e}^{-in\pi x/R'} \mathrm{d}x \qquad (3)$$

Spectral decomposition - Gaussian

E.g. Gaussian can be decomposed into monochromatic waves with amplitudes



Wavenumber and “uncertainty” relations

Relation between standard deviation of waveform and standard deviation of n - for Gaussian:

$$\sigma_n \sigma_x = \frac{R'}{\pi}$$

Can simplify (2) and (3) by defining wavenumber:

$$k_n \equiv \frac{n\pi}{R'}$$

Then “uncertainty relation” for Gaussian becomes

$$\sigma_k \sigma_x = 1$$

In general,

$$\sigma_k \sigma_x \geq 1$$

Classical FT $\longleftrightarrow$ QM part 1

Classical FT: $\sigma_k \sigma_x \geq 1$

QM: $\Delta k \, \Delta x \;\geq 1$

- equivalent to $\Delta p \, \Delta x \;\geq \hbar$ because $\Delta p = \hbar \Delta k$

Q) Can we define p for waveforms in classical FT? If so, what is the relation between p and k- is there a quantity corresponding to $\hbar$?

- we will return to this!

Monochromatic waves as orthonormal basis

- Monochromatic waves form an orthonormal basis for set of physically meaningful periodic waveforms
- Despite being a classical theory, Dirac's bracket notation is simplest way of representing this:

$$|n >= \mathrm{e}^{in\pi x/R'}$$

so

$$|\Phi >= \sum_{n} c_n |n >$$

…

Monochromatic waves as orthonormal basis

… then orthonormality relation is

$$< n|n' > \ = \begin{cases} 0 & n \neq n' \\ 1 & n = n' \end{cases}$$

so that inner product of Φ with another real waveform of period $2R$’ has the form

$$< \Phi|\Psi >= \sum_n c_n^* c_n'$$

Moving waveforms

Moving waveforms can also be decomposed into Fourier series

$$\Phi(x,t) = \sum_{n=-\infty}^{\infty} c_n \ \mathrm{e}^{ik_n x} e^{-i\omega_n (t-t_0)}$$

For example, for waveforms satisfying wave equation from action (1),

$$\Phi(x,t) = \sum_{n=-\infty}^{\infty} c_n \ \mathrm{e}^{ik_n x} \mathrm{e}^{-ik_n v(t-t_0)}$$

- *Note in relativistic scenario, for massless field*

$$(\omega_n)^2 - c^2 \sum_{\mathrm{j}=1}^{3} k_n^j k_{jn} = 0$$

Moving waveforms - bases

For moving waveforms:

- Can continue to use $|n>$, in which case time factor is contained in coefficients:

$$|\Phi(x,t)>=\sum_{n=-\infty}^{\infty} c_n\, e^{-i\omega_n \delta t}|n>$$

- OR can define moving basis

$$|n,\omega>=e^{-i\omega_n \delta t}|n>$$

so that

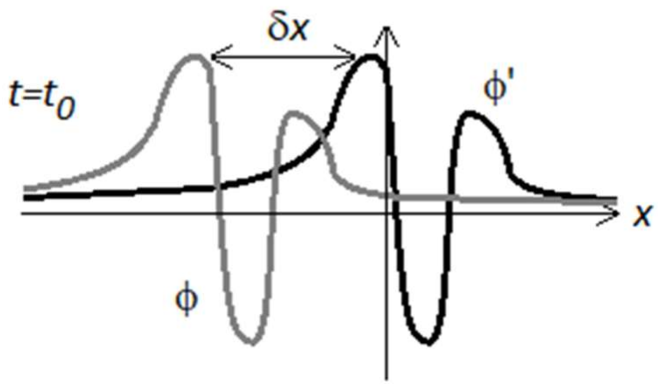
$$|\Phi(x,t)>=\sum_{n=-\infty}^{\infty} c_n\, |n,\omega>$$

Momentum – how to define

If we want something like Heisenberg's U.P. we will need to define momentum for our waveform

- Can't use $p = mv$ – only makes sense for particles
- Can't use $\{q\_i, p\_i\}$: on transition to FT, $\{q\_i, p\_i\} \longrightarrow \{\phi, \Pi\}$

Instead, use Noether's theorem. Displace field:



Displacing the field

(This is equivalent to translation of x coordinate, as described in notes)

It brings out another connection with QM: Taylor expanding the displaced field ϕ’ gives a power series in the derivative operator:

$$\phi'(x,t) = \phi(x,t) + (-\delta x)\frac{\partial\phi}{\partial x} + \frac{1}{2}(-\delta x)^2\frac{\partial^2\phi}{\partial x^2} + \frac{1}{3!}(-\delta x)^3\frac{\partial^3\phi}{\partial x^3} + \ \ldots$$

$$= \left(1 + (-\delta x)\frac{\partial}{\partial x} + \frac{1}{2}(-\delta x)^2\frac{\partial^2}{\partial x^2} + \frac{1}{3!}(-\delta x)^3\frac{\partial^3}{\partial x^3} + \cdots\right)\phi(x,t)$$

Displacing a basis state – a worthwhile digression

Want to see this for basis state $|n>$. The action of the operator on this state is

$$\frac{\partial|n>}{\partial x} = \mathrm{i}k_n|n>$$

Note similarity to eigenvalue eqn for momentum eigenstate in QM.

Calculate the powers, then subst. into Taylor series, noting power expansion of exponential – this gives us

$$|n>' = e^{-\mathrm{i}\,\delta x\,k_n}|n>$$

This is valid for ***all*** values of δx and k_n.

However, $|n>$ has fundamental period of $2\pi/k_n$. If δx is multiple of this, $|n>$ is invariant.

Noether procedure

Having looked at action of displacements on basis states, now return to Noether procedure for real field ϕ, focusing on solutions to wave eqn

Derivatives of displaced field are

$$\frac{\partial \phi'}{\partial t} = \frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial t \partial x}(-\delta x) + \mathcal{O}^2(\delta x)$$

and

$$\frac{\partial \phi'}{\partial x} = \frac{\partial \phi}{\partial x} + \frac{\partial^2 \phi}{\partial x^2}(-\delta x) + \mathcal{O}^2(\delta x)$$

Noether procedure (contd.)

Subst these into

$$\mathcal{L}' = \frac{1}{2}\rho\left(\frac{\partial \phi'}{\partial t}\right)^2 - \frac{1}{2}\rho v^2\left(\frac{\partial \phi'}{\partial x}\right)^2$$

& assume $\mathcal{L}' = \mathcal{L}$ to 1st order, to get continuity equation

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial x}\right) = \frac{\partial}{\partial x}\left[\rho v^2\left(\frac{\partial \phi}{\partial x}\right)^2\right]$$

- Ambiguity of (dimensionless) constant
- Requirement for δS to vanish for any δx over any space & time intervals
- Spatial integral of RHS is flux through endpoints of interval

Conserved momentum

For localised waveforms over large interval, flux is zero, thus

$$\frac{\partial}{\partial t}\left(\rho \frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial x}\right) = 0$$

Integrate conserved quantity over space to get conserved momentum:

$$p = \int_{-R}^{R} A\rho \frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial x}\ \mathrm{d}x = 0$$

where A is dimensionless constant

- Similar form to $< p >$ in QM:

$$< p >= \frac{1}{N}\int_{\text{all space}} \phi^* \hat{p}\ \phi\, \mathrm{d}x$$

- except expression for p has time derivative inside integral

Conserved momentum – stationary states and basis states

- This means that all stationary states have zero momentum – including stationary basis $|n>$
- Note that basis states in general are not localised (and are also complex) so shouldn't expect sensible result for momentum
- For moving basis, momentum would be proportional to integral of
$$|n>^2 = \mathrm{e}^{2ik_n x}$$
- This integral is zero over integer number of periods. Thus total momentum is zero if $R' = R$. Also zero in Fourier transform limit, $R' \to \infty$. Otherwise, momentum depends on amplitude of $|n>^2$ around boundary of universe $\longrightarrow$ pathologies

Conserved momentum – real waveforms

For our real, localised, physical waveform ϕ, we do get a sensible answer: with $R' = R$

$$p = -2RA\rho \sum_n c_n^* c_n \omega_n k_n$$

Very close to $< p >$ in QM:

$$< p >= \frac{\hbar}{N} \sum_n c_n^* c_n k_n$$

(where $N = \sum_n c_n^* c_n$). Difference in sum is ω_n. If ω were independent of n, sum would be the same.

How might we bridge the gap?

- Dependence of ω on n depends on equation of motion.
- For right-moving solutions of non-relativistic wave equation, $\omega_n = k_n v$.
- For massless relativistic field, we have

$$(\omega_n)^2 - c^2 \sum_{\mathrm{j}=1}^{3} k_n^j k_{jn} = 0$$

Looks like analysis for relativistic massless and massive fields would be worth exploring further. Would this result in something like

$$\hbar \propto \rho R \; ?$$

Can't know without doing analysis, but would a) possibly provide new interpretation of QM, b) have implications for higher-dimensional theories.

Summary

- Many of the features of QM are equally valid for classical field configurations
- Most of these can be found in textbooks, lectures etc, but not, as far as I know, assembled into a meaningful narrative:
  - Orthonormal basis states
  - Localised, physical waveforms described as superpositions of these
  - Uncertainty relation between position and wavenumber
- Monochromatic waves are eigenstates of the spatial derivative operator, and are invariant under translations which are a multiple of their fundamental period

Summary cont'd

- By using Noether's theorem, we can define momentum for classical field configurations – in a finite universe, this is finite and is meaningful for localised, physical waveforms, but not for basis states
- For these localised, physical waveforms, the expression for momentum is similar to that for the expectation value of momentum in QM
- However, for solutions of the non-relativistic wave equation, it is not quite close enough for us to identify a constant factor as $\hbar$ - but this is worth exploring for other actions

Questions?

Comments?

Reflections?