Presentation to CAMP, March 2021
My recent areas of physics research

• Applying coset space methods to spacetime symmetries
  • Compactification
    “Fully covariant spontaneous compactification” and RG posts
  • GR and teleparallelism
    “Tangent space symmetries in general relativity and teleparallelism”

• Roots of quantisation
  “Correspondence between Classical Field Theory in a finite universe and Quantum Mechanics – position, wavenumber and momentum”
Classical Field Theory $\leftrightarrow$ QM (in finite universe) – position, wavenumber, momentum

• Format is a set of notes, rather than a paper
• Much of it is standard theory for Classical Field Theory, but
  • Brought together from different sources
  • Applied to a finite universe (with 1 spatial dimension for simplicity)

appearance of quantum-like features slightly more cleanly than in standard theory, strong hints at dynamical interpretation of $\hbar$

• Questions:
  • Is there anything new here?
  • Are there flaws/holes in this analysis?
  • Is there anything that could be publishable in this?
Basic idea

• Eigenstates, superpositions and uncertainty relations usually seen as distinguishing features of QM

• Some authors see appearance of $\hbar$ as distinguishing feature, some just see $\hbar$ as scaling factor

BUT in classical field theory:

• Fourier analysis provides description of scalar field configuration as superposition of eigenstates of a derivative operator

• Uncertainty relations between position $x$ and wavenumber $k$

• Can in theory define momentum density for a field configuration
Basic idea (cont’d) – and sneak peek at results

THEREFORE:

• Try calculating momentum density and integrating over finite universe to get finite value of momentum $p$
• See if this gives us scaling factor between $p$ and $k$ dynamical interpretation of $\hbar$

• For simplicity, we do this with 1 spatial dimension, in non-relativistic situation
• Very nearly works – unclear whether it would work completely in relativistic case
Action and Euler-Lagrange equations

• Start with actional functional. Main one we consider is

\[ S = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \frac{1}{2} \rho \left( \frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} \rho v^2 \left( \frac{\partial \phi}{\partial x} \right)^2 \, dx \, dt \]  

(1)

• Leads to wave equation

\[ \frac{\partial^2 \phi}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 \phi}{\partial t^2} = 0 \]
Solutions

• Wide class of solutions – any function of form

\[ \phi = f_1(x - vt) + f_2(x + vt) \]

• Left-moving and right-moving parts

• Let \( x = 0 \) be centre of universe of radius \( R \). Consider solutions localised around \( x = 0 \) at time \( t = t_0 \).

• Can find spectral decomposition at \( t = t_0 \) by taking \( \phi \) to be part of waveform \( \Phi \) with period \( 2R' \geq 2R \).
Spectral decomposition

Fourier series

\[ \Phi = \sum_{n=-\infty}^{\infty} c_n \, e^{i n \pi x/R'} \]  \hspace{1cm} (2)

Where

\[ c_n = \frac{1}{2R'} \int_{-R'}^{R'} \Phi \, e^{-i n \pi x/R'} \, dx \]  \hspace{1cm} (3)
Spectral decomposition - Gaussian

E.g. Gaussian can be decomposed into monochromatic waves with amplitudes
Wavenumber and “uncertainty” relations

Relation between standard deviation of waveform and standard deviation of \( n \) - for Gaussian:

\[
\sigma_n \sigma_x = \frac{R'}{\pi}
\]

Can simplify (2) and (3) by defining wavenumber:

\[
k_n \equiv \frac{n\pi}{R'}
\]

Then “uncertainty relation” for Gaussian becomes

\[
\sigma_k \sigma_x = 1
\]

In general,

\[
\sigma_k \sigma_x \geq 1
\]
Classical FT $\leftrightarrow$ QM part 1

Classical FT:  $\sigma_k \sigma_x \geq 1$

QM:  $\Delta k \Delta x \geq \hbar$
- equivalent to $\Delta p \Delta x \geq \hbar$ because $\Delta p = \hbar \Delta k$

Q) Can we define $p$ for waveforms in classical FT? If so, what is the relation between $p$ and $k$? is there a quantity corresponding to $\hbar$?
- we will return to this!
Monochromatic waves as orthonormal basis

• Monochromatic waves form an orthonormal basis for set of physically meaningful periodic waveforms

• Despite being a classical theory, Dirac’s bracket notation is simplest way of representing this:

\[ |n \rangle = e^{in\pi x/R'} \]

so

\[ |\Phi \rangle = \sum_n c_n |n \rangle \]

...
Monochromatic waves as orthonormal basis

... then orthonormality relation is

$$< n|n' > = \begin{cases} 0 & n \neq n' \\ 1 & n = n' \end{cases}$$

so that inner product of $\Phi$ with another real waveform of period $2R'$ has the form

$$< \Phi|\Psi > = \sum_n c_n^* c'_n$$
Moving waveforms

Moving waveforms can also be decomposed into Fourier series

$$\Phi(x, t) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x} e^{-i\omega_n (t-t_0)}$$

For example, for waveforms satisfying wave equation from action (1),

$$\Phi(x, t) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x} e^{-ik_n v(t-t_0)}$$

- Note in relativistic scenario, for massless field

$$ (\omega_n)^2 - c^2 \sum_{j=1}^{3} k_n^j k_{jn} = 0 $$
Moving waveforms - bases

For moving waveforms:

• Can continue to use $|n>$, in which case time factor is contained in coefficients:

$$ |\Phi(x, t) > = \sum_{n=-\infty}^{\infty} c_n \, e^{-i\omega_n \delta t} |n> $$

• OR can define moving basis

$$ |n, \omega > = e^{-i\omega_n \delta t} |n> $$

so that

$$ |\Phi(x, t) > = \sum_{n=-\infty}^{\infty} c_n \, |n, \omega > $$
Momentum – how to define

If we want something like Heisenberg’s U.P. we will need to define momentum for our waveform

• Can’t use $p = mv$ – only makes sense for particles
• Can’t use $\{q_i, p_i\}$: on transition to FT, $\{q_i, p_i\} \rightarrow \{\phi, \Pi\}$

Instead, use Noether’s theorem. Displace field:
Displacing the field

(This is equivalent to translation of $x$ coordinate, as described in notes)

It brings out another connection with QM: Taylor expanding the displaced field $\phi'$ gives a power series in the derivative operator:

$$\phi'(x, t) = \phi(x, t) + (-\delta x) \frac{\partial \phi}{\partial x} + \frac{1}{2} (-\delta x)^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{3!} (-\delta x)^3 \frac{\partial^3 \phi}{\partial x^3} + \ldots$$

$$= \left(1 + (-\delta x) \frac{\partial}{\partial x} + \frac{1}{2} (-\delta x)^2 \frac{\partial^2}{\partial x^2} + \frac{1}{3!} (-\delta x)^3 \frac{\partial^3}{\partial x^3} + \ldots \right) \phi(x, t)$$
Displacing a basis state – a worthwhile digression

Want to see this for basis state $|n\rangle$. The action of the operator on this state is

$$\frac{\partial |n\rangle}{\partial x} = ik_n|n\rangle$$

Note similarity to eigenvalue eqn for momentum eigenstate in QM.

Calculate the powers, then subst. into Taylor series, noting power expansion of exponential – this gives us

$$|n\rangle' = e^{-i\delta x k_n}|n\rangle$$

This is valid for all values of $\delta x$ and $k_n$.

However, $|n\rangle$ has fundamental period of $2\pi/k_n$. If $\delta x$ is multiple of this, $|n\rangle$ is invariant.
Noether procedure

Having looked at action of displacements on basis states, now return to Noether procedure for real field $\phi$, focusing on solutions to wave eqn

Derivatives of displaced field are

$$\frac{\partial \phi'}{\partial t} = \frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial t \partial x} (-\delta x) + O^2(\delta x)$$

and

$$\frac{\partial \phi'}{\partial x} = \frac{\partial \phi}{\partial x} + \frac{\partial^2 \phi}{\partial x^2} (-\delta x) + O^2(\delta x)$$
Noether procedure (contd.)

Subst these into

\[ L' = \frac{1}{2} \rho \left( \frac{\partial \phi'}{\partial t} \right)^2 - \frac{1}{2} \rho v^2 \left( \frac{\partial \phi'}{\partial x} \right)^2 \]

& assume \( L' = L \) to 1\(^{\text{st}}\) order, to get continuity equation

\[ \frac{\partial}{\partial t} \left( \rho \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x} \right) = \frac{\partial}{\partial x} \left[ \rho v^2 \left( \frac{\partial \phi}{\partial x} \right)^2 \right] \]

- Ambiguity of (dimensionless) constant
- Requirement for \( \delta S \) to vanish for any \( \delta x \) over any space & time intervals
- Spatial integral of RHS is flux through endpoints of interval
Conserved momentum

For localised waveforms over large interval, flux is zero, thus

\[
\frac{\partial}{\partial t} \left( \rho \frac{\partial \Phi}{\partial t} \frac{\partial \Phi}{\partial x} \right) = 0
\]

Integrate conserved quantity over space to get conserved momentum:

\[
p = \int_{-R}^{R} A \rho \frac{\partial \Phi}{\partial t} \frac{\partial \Phi}{\partial x} \, dx = 0
\]

where \( A \) is dimensionless constant

- Similar form to \( \langle p \rangle \) in QM:
  \[
  \langle p \rangle = \frac{1}{N} \int_{\text{all space}} \Phi^* \hat{p} \Phi \, dx
  \]

- except expression for \( p \) has time derivative inside integral
Conserved momentum – stationary states and basis states

• This means that all stationary states have zero momentum – including stationary basis $|n>$

• Note that basis states in general are not localised (and are also complex) so shouldn’t expect sensible result for momentum

• For moving basis, momentum would be proportional to integral of $|n>^2 = e^{2i\kappa_n x}$

• This integral is zero over integer number of periods. Thus total momentum is zero if $R' = R$. Also zero in Fourier transform limit, $R' \to \infty$. Otherwise, momentum depends on amplitude of $|n>^2$ around boundary of universe → pathologies
Conserved momentum – real waveforms

For our real, localised, physical waveform $\phi$, we do get a sensible answer: with $R' = R$

$$p = -2RA\rho \sum_n c_n^* c_n \omega_n k_n$$

Very close to $<p>$ in QM:

$$<p> = \frac{\hbar}{N} \sum_n c_n^* c_n k_n$$

(where $N = \sum_n c_n^* c_n$). Difference in sum is $\omega_n$. If $\omega$ were independent of $n$, sum would be the same.
How might we bridge the gap?

• Dependence of \( \omega \) on \( n \) depends on equation of motion.
• For right-moving solutions of non-relativistic wave equation, \( \omega_n = k_n v \).
• For massless relativistic field, we have

\[
(\omega_n)^2 - c^2 \sum_{j=1}^{3} k^j_n k_{jn} = 0
\]

Looks like analysis for relativistic massless and massive fields would be worth exploring further. Would this result in something like

\[
\hbar \propto \rho R
\]

Can’t know without doing analysis, but would a) possibly provide new interpretation of QM, b) have implications for higher-dimensional theories.
Summary

• Many of the features of QM are equally valid for classical field configurations

• Most of these can be found in textbooks, lectures etc, but not, as far as I know, assembled into a meaningful narrative:
  • Orthonormal basis states
  • Localised, physical waveforms described as superpositions of these
  • Uncertainty relation between position and wavenumber

• Monochromatic waves are eigenstates of the spatial derivative operator, and are invariant under translations which are a multiple of their fundamental period
• By using Noether’s theorem, we can define momentum for classical field configurations – in a finite universe, this is finite and is meaningful for localised, physical waveforms, but not for basis states

• For these localised, physical waveforms, the expression for momentum is similar to that for the expectation value of momentum in QM

• However, for solutions of the non-relativistic wave equation, it is not quite close enough for us to identify a constant factor as $\hbar$ - but this is worth exploring for other actions