Using the coset formulation to examine the geometry of pseudo-Riemannian spacetimes

Presentation to CAMP, May 2021

My recent areas of physics research

- Applying coset space methods to spacetime symmetries
  - Compactification

    "Fully covariant spontaneous compactification" and RG posts
  - GR and teleparallelism

    "Tangent space symmetries in general relativity and teleparallelism"
- Roots of quantisation

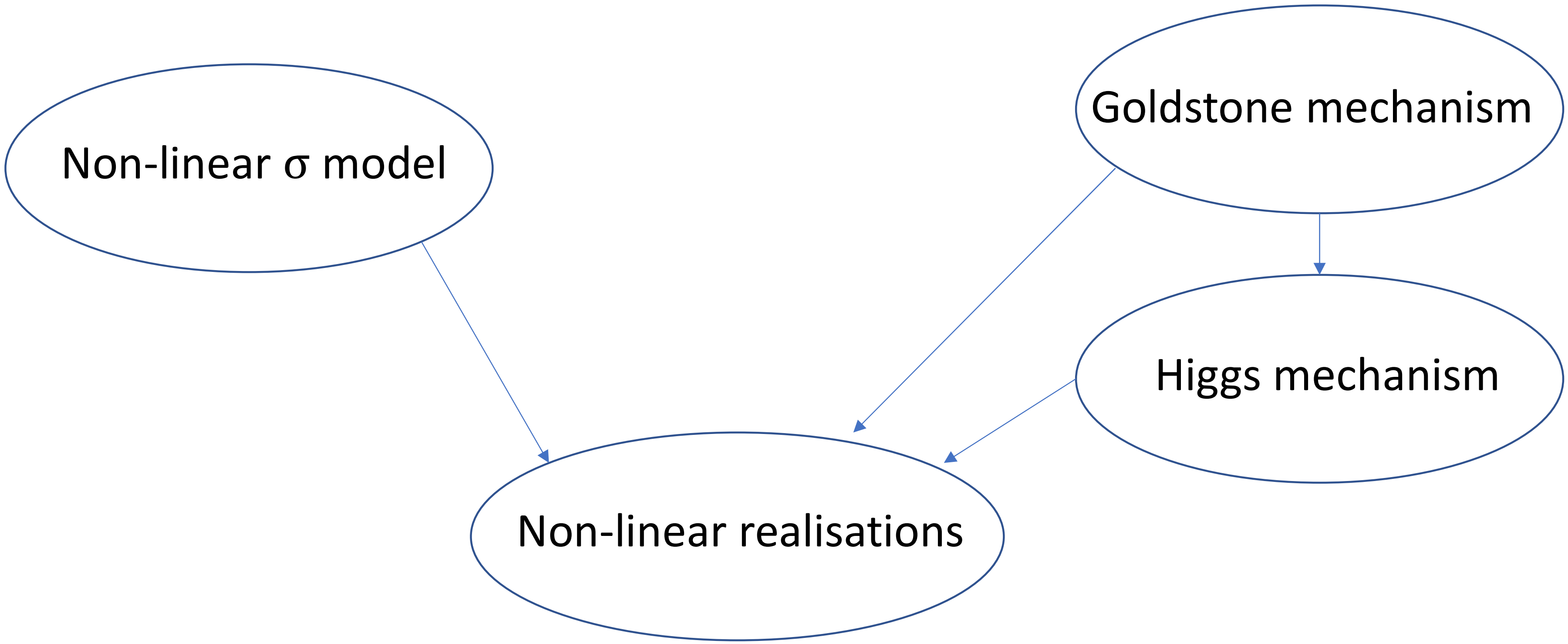
    "Correspondence between Classical Field Theory in a finite universe and Quantum Mechanics – position, wavenumber and momentum"

Plan of presentation

1. Non-linear realisations: using coset space methods for internal symmetries
2. GR and teleparallel gravity: using coset space methods for tangent space symmetries
3. Orbits & strata and solutions of gravitational field equations

Non-linear realisations

Methods developed in 1960s for internal symmetries



Key papers: (Callan), Coleman, Wess & Zumino: *Structure of Phenomenological Lagrangians I & II*, Phys Rev 177 ('69) 2239-50

Non-linear realisations cont'd

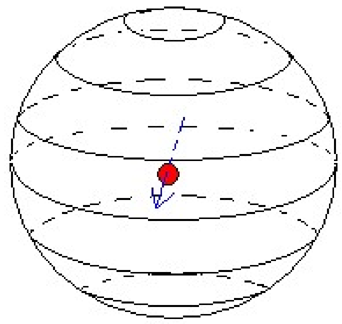
- Usually used in 1960s to describe:
  - Chiral symmetries: $(SU(N)_L \times SU(N)_R) / SU(N)_V$
  - Spherical field spaces: $SO(N+1) / SO(N) \equiv S^N$

We will illustrate using $SO(3) / SO(2) \equiv S^2$

- $G = SO(3)$; start with triplet of Lorentz scalars: ϕ^i
- Goldstone: apply Mexican hat potential: $V = a^2(\phi^i \phi_i - r^2)^2$
- V is min (V = 0) for $\phi^i \phi_i = r^2$, so vacuum manifold is sphere
- σ model: apply $\phi^i \phi_i = r^2$ as starting postulate

The two-sphere

For any chosen field state ϕ_0 on S^2, we can always define an axis passing through it:



This state is invariant under any H = SO(2) rotation about this axis:

$$H\,\phi_0 = \phi_0$$

Coset space methods

- H partitions G into cosets of the form gH where $g \in G$
- Consider rotations about z-axis, under which 'North Pole' is invariant

$$H = \{h = \mathrm{e}^{i\theta^3 T_3}\}$$

- Then gH has form

$$gH = \mathrm{e}^{i(\theta^1 T_1 + \theta^2 T_2)}\{h = \mathrm{e}^{i\theta^3 T_3}\}$$

- These cosets form 'coset space' with coordinates (θ^1, θ^2)
- Diffeomorphism between coset space and points on sphere given by

$$\phi = gH\phi_0 = \mathrm{e}^{i(\theta^1 T_1 + \theta^2 T_2)}\phi_0$$

where $\phi_0 = (0, 0, \mathrm{r})$ is the 'North Pole'.

- Thus (θ^1, θ^2) can be used as coordinates on sphere (embedding)

Goldstone interpretation

- θ^1, θ^2 are 'Goldstone bosons'
- Original triplet can be rewritten as $(\theta^1, \theta^2, \mathrm{r}')$ where r' is a radial field

$$\mathrm{r}' = |\phi| - r$$

Coset space representative and 'standard fields'

Each coset may be written in terms of a 'coset space representative'

$$L = \mathrm{e}^{i(\theta^1 T_1 + \theta^2 T_2)}$$

which has no subgroup generators in its exponent. Then if

$$g' \colon\ LH \ \rightarrow L'H \ = \ g'LH \ ,$$

L must transform as

$$g' \colon\ L \rightarrow g'L = L'h'$$

Its inverse, L^{-1}, may be used to rewrite all multiplets of G as multiplets of H only:

$$\psi = L^{-1}\Psi$$

Then easy to show that

$$g' \colon \psi \rightarrow \psi' = h'\psi$$

Now we want to see how we can use these coset methods to study theories of gravity…

Tangent Space Symmetries in General Relativity and Teleparallelism
https://doi.org/10.1142/S0219887821400089

General relativity

GR isn't **just*

$$S = \int_\Omega (-g)^{\frac{1}{2}} R + k\mathcal{L}_\mathrm{M}\, \mathrm{d}\Omega$$

Misses key points:

- $m_\mathrm{I} = m_\mathrm{G}$ $\longrightarrow$ Equivalence principle & LIFs
- $\longrightarrow$ Test particles moving on geodesics; geodesic equation
- Neighbouring geodesics $\longrightarrow$ gravity = curvature

LIFs: in limit that gravity/acceleration can be neglected, spacetime reduces to Minkowski spacetime: pseudo-Riemannian

More general gravitational theories

- Generally start with a different action, e.g. f(R), f(T),…
- BUT usually describe gravity as geometric property. If they do not reduce to Minkowski spacetime in appropriate limits (or provide equivalent results), they are pure maths – THEY DO NOT REPRESENT THE REALITY WE OBSERVE
- A lot can be deduced purely from this requirement, without postulating action/field equations

Metric and tetrads

In original formulation of GR, each coordinate system has a metric field associated with it. Under changes of coordinates, this transforms according to

$$g_{\mu\nu}^{(u')} = \frac{\partial u^{\rho}}{\partial u'^{\mu}} \frac{\partial u^{\lambda}}{\partial u'^{\nu}} g_{\rho\lambda}^{(u)}$$

In tetrad formulation, metric at a point A is viewed as inner product of basis vectors in $T_A\mathcal{M}$:

$$g_{\mu\nu}\Big|_A = \left(\boldsymbol{e}_{\mu}, \boldsymbol{e}_{\nu}\right)_A$$

which transform according to

$$\boldsymbol{e}_{\mu}^{(u')}\Big|_A = \frac{\partial u^{\nu}}{\partial u'^{\mu}}\bigg|_A \boldsymbol{e}_{\nu}^{(u)}\Big|_A$$

Tetrads cont'd

Often, the breakdown of $\boldsymbol{e}_{\mu}^{(u')}$ in the u-coordinate system is written

$$\boldsymbol{e}_{\mu}^{(u')}\Big|_{A} = e'_{\mu}{}^{\nu}\ \boldsymbol{e}_{\nu}^{(u)}\Big|_{A}$$

In particular, at any point A we can choose a frame basis $\widehat{\boldsymbol{n}}_{\mu}$ with inner product

$$\left(\widehat{\boldsymbol{n}}_{\mu}, \widehat{\boldsymbol{n}}_{\nu}\right)_{A} = \eta_{\ \mu\nu}$$

so that

$$\boldsymbol{e}_{\mu}^{(u)}|_{A} = e_{\mu}{}^{\nu}\,\widehat{\boldsymbol{n}}_{\nu}|_{A}$$

Then all the d.o.f. are carried in $e_{\mu}{}^{\nu}$ (which is often referred to as the tetrad)

Connections

Levi-Civita:

- Used in original formulation of GR
- Constructed from metric
- Metric-compatible
- Symmetric on lower indices
- $\longrightarrow$ For a given coordinate system, uniquely defined across coordinate neighbourhood
- BUT parallel transport along segments of different geodesics don't commute – so result depends on path

Connections cont'd

If you are parallel transporting a vector along a complicated path and you want to avoid having to combine lots of sections, you need p.t. which is independent of path – just determined by location.

This corresponds to the **Weitzenböck connection**:

- Constructed from tetrad components
- Metric-compatible
- Not symmetric on lower indices: torsion
- BUT for a given coordinate system, NOT unique – depends on frame, as we shall see
- HOWEVER, once coordinates and frames are chosen, it is uniquely defined – and parallel transport is independent of path taken

Teleparallel gravity

- A theory of gravity which uses the Weitzenböck connection is known as a teleparallel theory
- It has been shown that GR can be formulated as a teleparallel theory: TEGR (with action built from torsion tensor) has same field equations
- Other teleparallel theories have been put forward, where the action is built from the torsion tensor in other ways

Lorentz gauge transformations

- Some changes of frame affect the value of the Weitzenböck connection, but not the metric
- Consequently, GR is invariant under these
- Under these changes of frame, spin connection (associated with Weitzenböck connection) transforms as a gauge potential
- Spin connection can be eliminated through an appropriate choice of frame field. It became commonplace for teleparallel gravity theorists to work in 'Weitzenböck gauge'
- However, choice of frame affects solutions of field equations for f(T) theories – "good" and "bad" tetrads!
- Many researchers associate changes of frame with inertial effects – incorrectly, as shown in paper

Result – confusion!

- Different researchers using different terminology for the same quantities
- Different researchers using different symbols for the same quantities
- Different researchers using the same symbols for different quantities

Underlying problem:

- 'Weitzenböck gauge' is not consistent with general covariance, as we shall see…

Coset formulation

My approach is to replace:

- tetrad formulation – which isn’t well-suited to teleparallel gravity,

with:

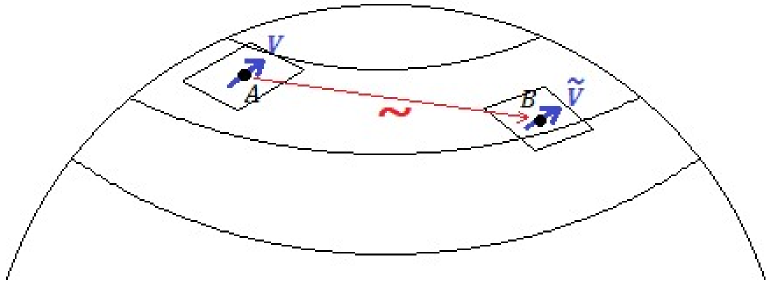
- ‘coset formulation’ – which separates out the Lorentz gauge d.o.f. from the metric d.o.f. in a natural way.

Coset formulation:

- Takes techniques from the method of non-linear realisations
- Applies them to changes of basis on a *single* tangent space
- Uses parallel transport to knit transformations together into fields

Parallel transport: parallel maps

Parallel transport as map $\sim$ between tangent spaces:

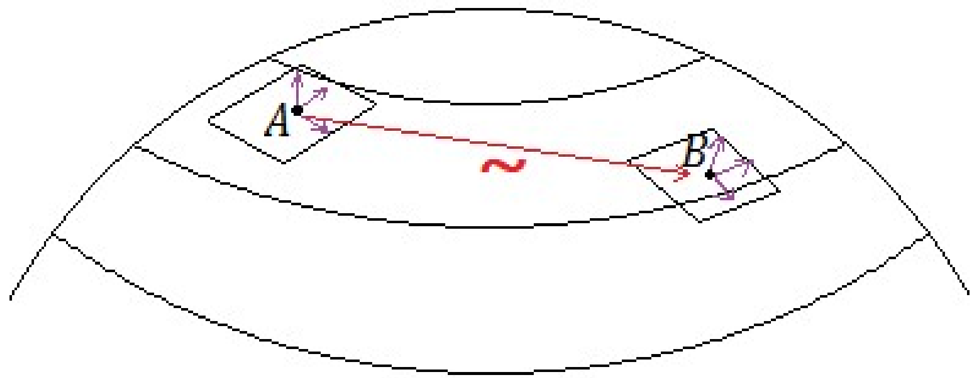


A metric-compatible linear connection is one which is associated with a parallel map which:

- Acts linearly on vectors: $\sim : \ \alpha\boldsymbol{V} + \beta\mathbf{W} \ \rightarrow \ \alpha\widetilde{\boldsymbol{V}} + \beta\widetilde{\boldsymbol{W}}$
- Preserves the inner product: $\sim : (\boldsymbol{V}, \mathbf{W}) \rightarrow \left(\widetilde{\boldsymbol{V}}, \widetilde{\boldsymbol{W}}\right)$

Parallelisms

This means that a frame basis is always mapped to another frame basis:



- Extend $\sim$ to 'parallelism' across coordinate neighbourhood, by choosing image of $\hat{\boldsymbol{n}}_{\mu}|_{A}$ on every tangent space to neighbourhood
- Can't generally be done for whole manifold (but not necessary for analysis)

Geometric meaning of Lorentz gauge transformations

- Need parallelism to be continuous, to define connection
- $\longrightarrow$ Frame field
- Any frame field can be used to define a Weitzenböck connection
- Frame fields are related by Lorentz gauge transformations

Bases on $T_A\mathcal{M}$

Each frame basis at A is a basis for a set of Riemann normal coordinates x^{ρ}

- recall that 2 coord bases are related by

$$\boldsymbol{e}_{\mu}^{(u')}\Big|_A = \left.\frac{\partial u^{\nu}}{\partial u'^{\mu}}\right|_A \boldsymbol{e}_{\nu}^{(u)}\Big|_A$$

Thus $\hat{\boldsymbol{n}}_{\nu}|_A$ and $\boldsymbol{e}_{\mu}^{(u)}|_A$ are related by

$$\boldsymbol{e}_{\mu}^{(u)}\Big|_A = \left.\frac{\partial x^{\nu}}{\partial u^{\mu}}\right|_A \hat{\boldsymbol{n}}_{\nu}\Big|_A$$

Functions *versus* values; J_{A} and I_{A}

- E.g. $(u^0, u^1, u^2, u^3) = (3(u'^0)^2, u'^1 + u'^2, u'^2 - (u'^3)^2, -15u'^1)$
- Then each $\frac{\partial u^{\nu}}{\partial u'^{\mu}}$ is a function, e.g. $\frac{\partial u^2}{\partial u'^3}$ = -2 u'^3
- But each $\left.\frac{\partial u^{\nu}}{\partial u'^{\mu}}\right|_A$ is a value, e.g. if $u'^3 = 7$ at A, $\left.\frac{\partial u^2}{\partial u'^3}\right|_A$ = -14
- Thus $\left.\frac{\partial u^{\nu}}{\partial u'^{\mu}}\right|_A$ is an invertible matrix of real *numbers*
- These form a group, $J_A \cong \mathrm{GL}(4, \mathrm{R})$
- The Jacobian matrices $\left.\frac{\partial x^{\nu}}{\partial x'^{\mu}}\right|_A$ which relate different *frame* bases form a subgroup, $I_A \cong \mathrm{O}(1,3)$

Coset decomposition of j_0

Therefore *choose* frame basis $\hat{n}_\nu|_A$ (this will be used for constructing parallelism) and write

$$\boldsymbol{e}^{(u)}_\mu|_A = j_0|_A \hat{\boldsymbol{n}}_\nu|_A$$

where

$$(j_0)_\mu{}^\nu|_A = \left.\frac{\partial x^\nu}{\partial u^\mu}\right|_A \in J_A$$

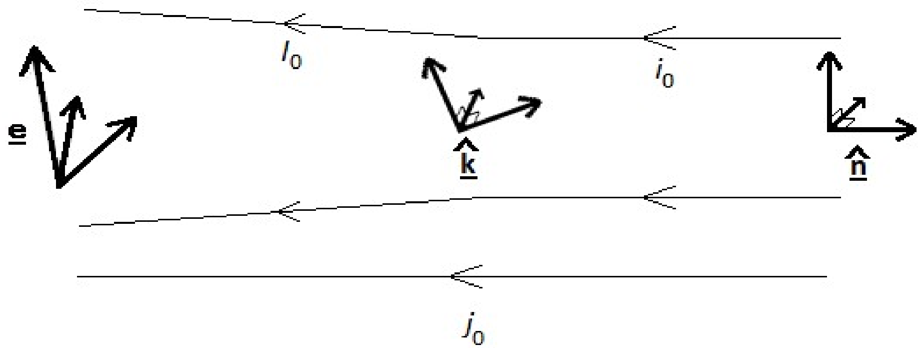
Then $j_0|_A$can uniquely be decomposed in form

$$j_0|_A = l_0|_A \, i_0|_A$$

where $i_0|_A \in I_A$ and $l_0|_A$ is a representative of J_A/I_A with no generators of I_A in its exponent

Visualising this decomposition & separating d.o.f.

- If $\widehat{\boldsymbol{n}}$ is frame basis associated with parallelism, then $\widehat{\boldsymbol{k}} = i_0\,\widehat{\boldsymbol{n}} = l_0^{-1}\boldsymbol{e}$ is also a *frame* basis, ‘intermediate’ between $\widehat{\boldsymbol{n}}$ and $\boldsymbol{e}$:



- Metric is then

$$g^{(u)}_{\mu\nu}\Big|_A = (l_0)_\mu{}^\rho\Big|_A (l_0)_\nu{}^\lambda\Big|_A \eta^{(u)}_{\rho\lambda}$$

- Thus l_0 carries metric d.o.f.; i_0 carries parallelism d.o.f
- Then change of frame (parallelism) acts on j_0 from right; change of coords acts from left

Extend to coordinate neighbourhood (chart)

- Can now extend to coordinate neighbourhood (with i_0 in same connected component everywhere)
- *BUT on curved spacetime, $\widehat{\boldsymbol{n}}_\nu$ is not a *coordinate* basis everywhere*
- Weitzenböck connection can easily be shown to take the forms

$$\overset{\bullet}{\Gamma}_{\lambda\nu}{}^{\mu}(\mathrm{u}) \equiv -(j_0 \partial_\lambda j_0^{-1})_\nu{}^{\mu} = -(l_0 \partial_\lambda l_0^{-1})_\nu{}^{\mu} + (l_0 (i_0 \partial_\lambda i_0^{-1})\, l_0^{-1})_\nu{}^{\mu}$$

- Have now seen how formulation gives geometric picture of Lorentz gauge transformation
- Now want to look at how gauge relates to coordinate choices…

Reason for ‘good’ and ‘bad’ tetrad problem

Action of j' on j_0

$$j' \colon j_0 \to j' j_0 = j' l_0 i_0$$

but from coset theory

$$j' l_0 = l' i'$$

- it affects both coset factor and subgroup factor, thus changing Lorentz gauge (spin connection)

So Weitzenböck gauge is coordinate-dependent – *it is not compatible with general covariance*

Inertial effects

- This formulation provides a helpful framework/background for considering inertial effects
- This is explained in paper…
- …but key point is that moving to e.g. rotating reference frame is a change of coords – affecting l as well as i
- Changes of i alone cannot be felt by observer, as they are not metric degrees of freedom

Extending the theory to study solutions of gravitational field equations…

Orbits of GL(N,R) and product manifolds
(in preparation)

Action by conjugation on tensors in Lie algebra

Key insights:

- j: $X_{\mu}^{(u)\nu} \rightarrow X_{\mu}^{(u')\nu} = (j\, X^{(u)} j^{-1})_{\mu}{}^{\nu}$
- $X_{\mu}^{(u)\nu} \in \mathcal{j}_A \triangleq \mathrm{gl}(4, \mathrm{R})$ or, more generally, $\mathrm{gl}(\mathrm{N}, \mathrm{R})$

Action of group by conjugation on its own algebra already researched for internal symmetries:

- Michel & Radicati: *The geometry of the octet*, Ann. Inst. Henri Poincaré XVIII ('73) 185-214: primarily SU(3)
- Extended to other SU(N) and SO(N) groups by me and other authors

Orbits and their invariants

- Action by conjugation preserves eigenvalues, partitioning algebra into 'orbits'
- Eigenvalues determined by invariants in characteristic equation
- For $\mathrm{gl}(\mathrm{N}, \mathrm{R})$ these are tr(X), tr(X^2), tr(X^3), …
- For $R_{\mu}{}^{\nu}$ in GR, these are first four Carminati-McLenaghan invariants

Cartan subspaces of su(N)

- For su(N), each orbit contains at least one diagonal matrix
- Set of all diagonal matrices form Cartan subalgebra, e.g. for SU(4) ≈ SO(6)

$$C_d = < T_3, T_8, T_{15} > = < \sigma_{12}, \sigma_{34}, \sigma_{56} >$$

- Relations (e.g. commutation) in C_d preserved under conjugation

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Matrices with distinct eigenvalues invariant only under group elements generated by Cartan subalgebra, e.g.

$$\begin{pmatrix} 2 & & & \\ & 1 & & \\ & & 3 & \\ & & & 5 \end{pmatrix} \text{ stabilised by } U(1) \otimes U(1) \otimes U(1)$$

Matrices with repeated eigenvalues have larger stabiliser groups $\longrightarrow$ ‘strata’

Strata and their stabilisers for $\mathrm{gl}(\mathrm{N},\mathrm{R})$

Strata in $\mathrm{gl}(\mathrm{N},\mathrm{R})$ are stabilised by products of GL groups, e.g.

$$\begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 3 \end{pmatrix} \text{stabilised by } \mathrm{GL}(2,\mathrm{R}) \otimes \mathrm{GL}(2,\mathrm{R}) \otimes \mathrm{GL}(1,\mathrm{R})$$

- The Ricci tensor and Einstein tensor – and in GR, the EMDT – have the same stabiliser
- This gives us a different coset decomposition: $j = L\, g$ where $\mathrm{g} \in \mathrm{Stab}(R_{\mu}{}^{\nu})$

This stabiliser gives us information about the local shape of the spacetime:

- A product space – one with block diagonal metric in appropriate coords – has more than one GL factor in its stabiliser
- For a Cartesian product space, the metric-preserving subgroup of the stabiliser is the holonomy group of the Levi-Civita connection

Summary

- Frame basis at any point is a coord basis for Riemann normal coords
- Allows coset space methods in non-linear realisations to be adapted for tangent space symmetries
- Constructing parallelisms gives geometric interpretation of Lorentz gauge transformations of spin connection
- Together, these constitute ‘coset formulation’ – appropriate to teleparallel gravity, separating d.o.f.
- Shows that setting spin connection to zero and changing coordinates are not separate procedures

Summary cont'd

- Theory of orbits and strata, developed for su(N), can also be adapted to tangent space symmetries
- Values of rank-2 tensors form orbits under changes of basis, distinguished by eigenvalues/invariants
- Stabiliser of Ricci/Einstein tensor provides info about local shape of space

Questions?

Comments?

Reflections?