Covariant compactification and unification

Presentation to CAMP, May 2022

My recent areas of physics research

- Applying coset space methods to spacetime symmetries
  - Compactification

    “Fully covariant spontaneous compactification”
    and RG posts

  - GR and teleparallelism

    “Tangent space symmetries in general relativity and teleparallelism”

- Roots of quantisation

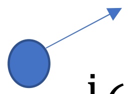
  “Correspondence between Classical Field Theory in a finite universe and Quantum Mechanics – position, wavenumber and momentum”

Overview of talk

- Particles, fields and rotations in classical, relativistic and quantum mechanics
- General relativity – reminder of key points
- Gauge fields
- Non-linear realisations: my introduction to physics research
- Covariant compactification – motivation and aim of programme
- Covariant compactification – eigenvalues, product manifolds and gauge fields
- Covariant compactification – (first) field equation
- Covariant compactification – Spinors and SU(3)
- Covariant compactification – linear coordinate transformations and field configurations
- Covariant compactification – developing field equations

Particles, fields and rotations in classical, relativistic and quantum mechanics

Particles and fields in classical mechanics



$x^{i}(t)$

Particles obey Newton's 2nd law:

$$\frac{\mathrm{d}p^{i}}{\mathrm{d}t} = F^{i}$$

Forces $\longleftrightarrow$ potentials, e.g.

$$F^{i} = -m\partial^{i}\phi$$

Potentials are fields – they obey field equations, e.g.

$$\nabla^{2}\phi = 4\pi G$$

In CM, $t, x^{1}, p^{1}, F^{1}, \ldots, m, \phi, G \in \mathbb{R}$

Scalars, vectors and rotations

m, ϕ and G are scalars:

- Single valued at $x^{\mathrm{i}}(t)$
- Value invariant under rotations

F^{i}, p^{i} and $\partial^{\mathrm{i}}\phi$ are vectors:

- One component for each dimension
- Components transformed into each other under rotations
- Vector has a magnitude – a scalar – invariant under rotations

Special orthogonal groups

Definitions:

- Orthogonal matrix:

$$\mathbf{o}^{T}\mathbf{o} = \mathbf{1}$$

- Special matrix:

$$|\mathbf{s}| = 1$$

For a vector rotation:

$$V'^{\mathrm{i}} = R^{\mathrm{i}}{}_{\mathrm{j}} V^{\mathrm{j}}$$

$R^{\mathrm{i}}{}_{\mathrm{j}}$ is both special and orthogonal.

- Such matrices form group SO(N)

Quantum mechanics

- In QM, $x^{\mathrm{i}}(t) \longrightarrow \psi(x^{\mathrm{i}}, t)$ where $\psi \in \mathbb{C}$
- Satisfies

$$\hat{H}\psi = \hat{E}\psi$$

where

$$\hat{E} = i\hbar \frac{\partial}{\partial t}$$

and typically

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x^{\mathrm{i}}, t)$$

Four-vectors

- Minkowski:

$$V^{\mu} = (V^{0}, V^{i})$$

- Rotations & boosts mix these up:

$$V'^{\mu} = L^{\mu}{}_{\nu} V^{\nu}$$

though

$$V'^{\mu} V'_{\mu} = V^{\mu} V_{\mu}$$

- Such matrices form group SO(1,3)
- In a more general spacetime with *t* time dimensions and *s* space dimensions

SO(1,3) $\longrightarrow$ SO(*t*,*s*)

Relativistic quantum mechanics - spin

SR + QM $\longrightarrow$ spin

This can be used to classify fundamental particles:

Spin	**Examples**
0	Higgs
½	Electron, quark
1	Photon, gluon

Relativistic quantum mechanics – field equations

- Schrödinger's equation – not relativistic
- Relativistic QM: complex wavefunctions $\longrightarrow$ complex fields – these vary with the spacetime coordinates x^{μ}
- For spin zero (scalar) fields, Schrödinger's equation $\longrightarrow$ Klein-Gordon equation:

$$\partial_{\mu}\partial^{\mu}\phi + m^2\phi = 0$$

- For spin ½ fields, Schrödinger's equation $\longrightarrow$ Dirac equation:

$$(\mathrm{i}\boldsymbol{\gamma}^{\mu}\partial_{\mu} - m\mathbf{1})\psi = 0$$

Quantum field theories

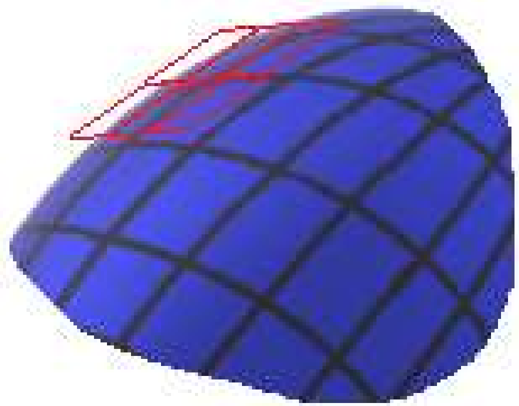
In QFTs, a fundamental particle is seen as an **excitation of the corresponding field**:

<u>**Particle**</u>		<u>**Field**</u>
Electron	$\longleftrightarrow$	Electron field
Quark	$\longleftrightarrow$	Quark field
Photon	$\longleftrightarrow$	Electromagnetic field
.		.
.		.
.		.

General relativity – reminder of key points

Tangent spaces

- GR: spacetime is curved (pseudo-Riemannian) manifold.
- Analysed using vectors in tangent spaces



- Scalar product on a tangent space: $(\boldsymbol{V}, \boldsymbol{W})$

Vector fields and connections

- Use “bundle” of tangent spaces to construct vector fields
- Coefficients of scalar product are metric tensor:

$$(\boldsymbol{V}, \boldsymbol{W}) = g_{\mu\nu} V^{\mu} W^{\nu}$$

- Use connection to compare values of a vector field in different tangent spaces and calculate $D_{\nu} V^{\mu}$
- Infinite choice of connections; GR: Levi-Civita connection

Riemann and Ricci curvature tensors

- Riemann tensor: field strength of the Levi-Civita connection

$$R^{\mu}{}_{\kappa\lambda\nu} = \partial_{\lambda}\Gamma_{\nu\kappa}{}^{\mu} - \partial_{\nu}\Gamma_{\lambda\kappa}{}^{\mu} + \Gamma_{\nu\kappa}{}^{\rho}\Gamma_{\lambda\rho}{}^{\mu} - \Gamma_{\lambda\kappa}{}^{\rho}\Gamma_{\nu\rho}{}^{\mu}$$

- Ricci tensor:

$$R_{\kappa\nu} = R^{\mu}{}_{\kappa\mu\nu}$$

- GR: $R_{\kappa\nu} = 0$ in any matter-free region
- Remainder of Riemann tensor $\neq 0$ in vicinity of matter

  – this quantifies curvature $\longrightarrow$ gravitational field

GR field equation

- GR: matter distribution $\longrightarrow T_{\mu\nu}$
- The field equation tells us how the energy-momentum density of matter at a given point relates to the curvature at that point:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\lambda} R_{\rho\lambda} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Transformation under changes of coordinates

- Vectors and tensors have simple transformation laws – e.g.

$$R'_{\mu\nu} = j_\mu{}^\kappa \, j_\nu{}^\lambda R_{\kappa\lambda}$$

  where

$$j_\mu{}^\kappa = \frac{\partial u^\kappa}{\partial u'^\mu}$$

  is an invertible, real matrix

- Such matrices form group GL(4, $\mathbb{R}$)

Gauge fields

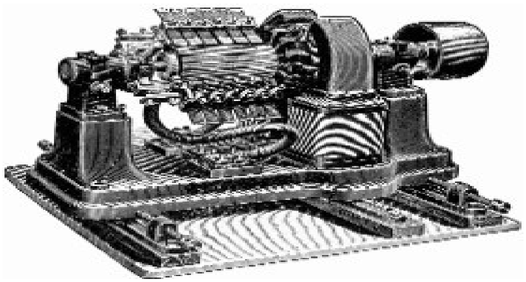
Electricity and magnetism

Both electrical and magnetic phenomena recognised for centuries:

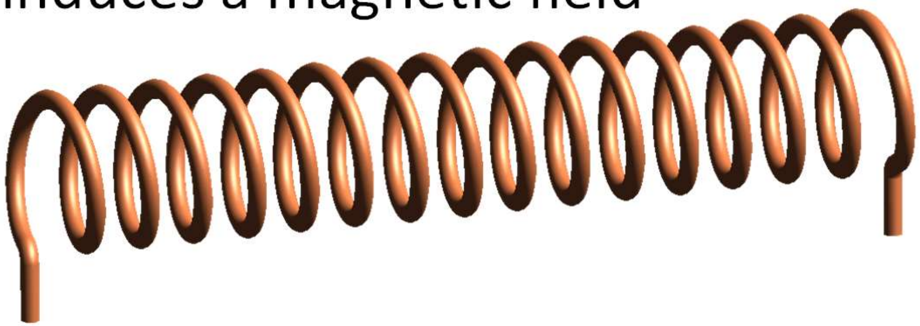
- Electricity – lightning and static
- Magnetism – magnets and lodestones/compasses

But not separate phenomena:

- In a dynamo, a magnetic field induces an electric current

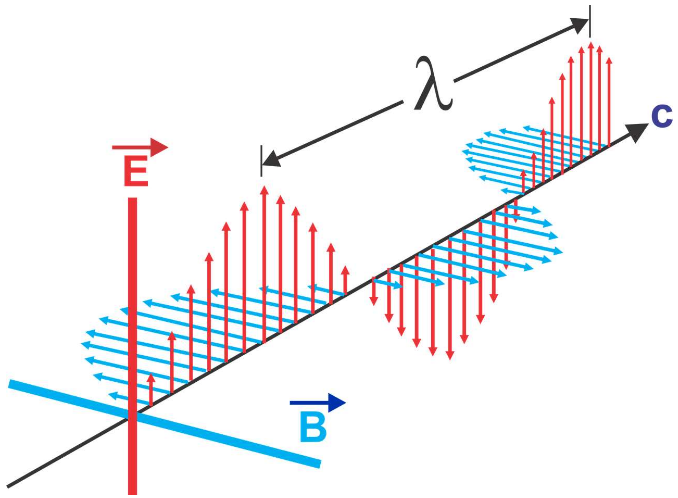


- In a solenoid, an electric current induces a magnetic field



Electromagnetism

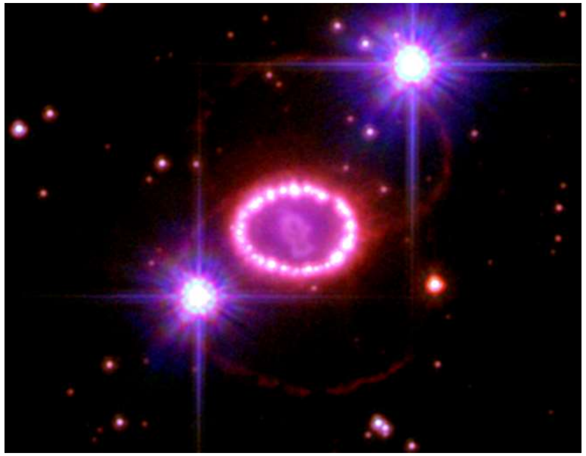
Electromagnetism



- an example of ‘unification’

Weak interaction and electroweak force

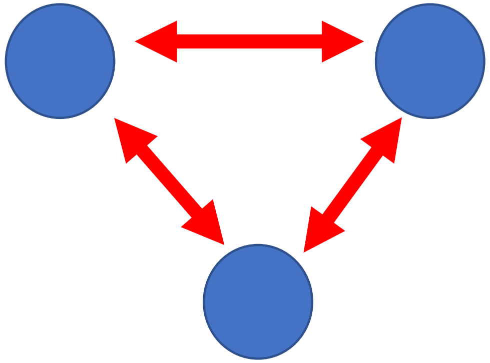
Weak interaction – radioactive β decay, fusion, supernovae



- Not separate from electromagnetism – unify at high energies: **electroweak force**
- Separate out at lower energies because of ‘symmetry breaking’ – see later

Strong force

A binding force:



- Binds quarks together → protons and neutrons
- Then binds protons and neutrons together → atomic nuclei
- Like electroweak force, described by a **quantum field theory**
- QFTs describe forces as mediated by **gauge fields** of **unitary symmetries** – will explore this later

Grand Unified Theories

- **Grand Unified Theories** unify electroweak + strong
  – at high energies
- No unique GUT
- Simplest ones similar to electroweak unification – but predict proton decay:
    - not observed
    - predicted to be more frequent than consistent with experimental evidence

Special unitary groups

Definition - unitary matrix:

$$\mathbf{u}^{\dagger}\mathbf{u} = \mathbf{1}$$

- Unitary transformation of *N*-dimensional complex vector

$$\psi'^{i} = u^{i}{}_{j}\psi^{j}$$

  looks like rotation – it preserves orthonormality on complex vector space

- Such matrices form group U(*N*)
- U(1) is composed of all complex numbers with $|z| = 1$ – changes of phase
- U(*N*) has SU(*N*) subgroup: matrices which are both special and unitary

Gauge fields - electromagnetism

- Quantum wave equations (Schrödinger, Klein-Gordon, Dirac) are invariant under constant phase factor…
- …but not under 'local' (varying) phase factor $\alpha(x^{\mu})$
- To make them invariant under 'local' change of phase, couple the field to a U(1) gauge field:

$$\partial_{\mu} \rightarrow D_{\mu} = \partial_{\mu} + iqA_{\mu}$$

where A_{μ} transforms according to:

$$A_{\mu} \rightarrow A'_{\mu} = A_{\mu} + \frac{1}{q}\partial_{\mu}\alpha$$

- Maxwell gauge invariance $\longrightarrow$ identify U(1) gauge field with EM potential

Gauge fields – electroweak

Now take two complex fields ψ^1 and ψ^2, each satisfying its own wave equation

- Quantum wave equations – invariant under ‘global’ U(1) and SU(2) transformation

$$\begin{matrix}\psi^1 \\ \psi^2\end{matrix} \xrightarrow{\text{U(1)}} \begin{matrix}\psi'^1 = e^{i\alpha}\psi^1 \\ \psi'^2 = e^{i\alpha}\psi^2\end{matrix}$$

$$\begin{matrix}\psi^1 \\ \psi^2\end{matrix} \xrightarrow{\text{SU(2)}} \begin{matrix}\psi'^1 = u^1{}_1\psi^1 + u^1{}_2\psi^2 \\ \psi'^2 = u^2{}_1\psi^1 + u^2{}_2\psi^2\end{matrix}$$

- Can make them invariant under local SU(2) and U(1) by coupling to a set of 3 fields with appropriate transformation properties

- Gauge invariance $\longrightarrow$ identify SU(2) and U(1) gauge fields with electroweak interaction

Gauge fields – strong force

Similarly, for 3 complex fields – we can ensure local SU(3) invariance by coupling to SU(3) gauge field:

$$\partial_\mu \begin{pmatrix} \psi \\ \psi \\ \psi \end{pmatrix} \rightarrow D_\mu \begin{pmatrix} \psi \\ \psi \\ \psi \end{pmatrix} = \partial_\mu \begin{pmatrix} \psi \\ \psi \\ \psi \end{pmatrix} + iqA_\mu^\alpha\, T_\alpha \begin{pmatrix} \psi \\ \psi \\ \psi \end{pmatrix}$$

where T_α are generators of SU(3) and A_μ^α =  transforms according to:

$$A_\mu^\alpha T_\alpha \rightarrow A_\mu'^\alpha T_\alpha = uA_\mu u^{-1} + \frac{1}{q} u\partial_\mu u^{-1}$$

Key features of the fundamental forces

Gravity

- Described by a geometric field theory
- Relates to spacetime transformations (see T. Lawrence, *Tangent space symmetries in general relativity and teleparallelism,* Int. J. Geom. Meth. Mod. Phys. 18 (2021) supp01, 2140008, doi:10.1142/S0219887821400089)
- Theory is in agreement with observations
- Within the heart of black holes, calculations result in infinities

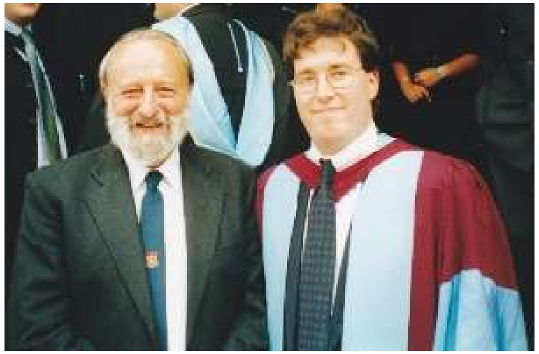
Electroweak force, strong force

- Described by quantum field theories
- Theories of gauged unitary symmetries
- Calculating properties of particles and their interactions results in infinities
- Mathematical procedures can remove infinities
  - Then results of theories are in agreement with observations
  - But physical meaning of these procedures is unclear

Non-linear realisations: my introduction to physics research

My PhD

- Southampton, England: 1997 – 2001, under Prof. Ken Barnes

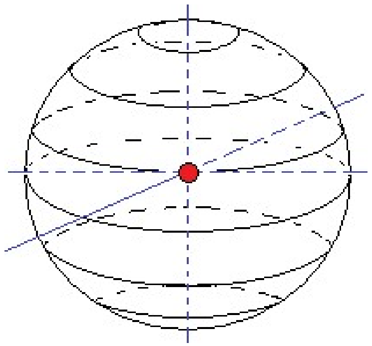


- On non-linear realisations of non-gravitational symmetries

Non-linear realisations

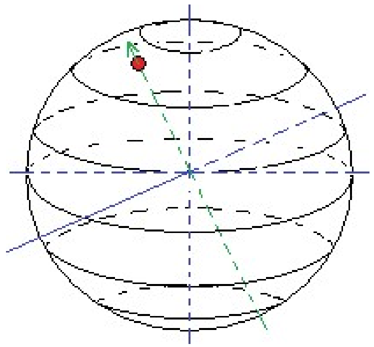
Non-linear realisations result when a symmetry is spontaneously broken. Consider a bead inside a spherical shell.

If bead is in centre of the shell:



then system is symmetric under 3D rotations: SO(3)

If bead is resting against shell itself:



then system is only symmetric under 2D rotations, about an axis passing through the bead: $SO(2) \subset SO(3)$

- *see* T Lawrence, *Non-linearly realised O(3) symmetries, ResearchGate*

Non-linear realisations

- Say the bead is at the 'North pole' – then system is symmetric under R_z
- Any other point can be reached from this by combining an R_x with an R_y
- R_x, R_y parametrised by θ^1 and θ^2 - which can therefore be used as coordinates for the spherical surface

Action of rotations on these coordinates:

- transform linearly under SO(2)
- transform non-linearly under rest of SO(3)

σ-models & spontaneous symmetry breaking

Analogous situation in field theory – the SO(3) σ-model:

- take ϕ^1, ϕ^2 and ϕ^3 which transform as vector under SO(3)
- These form a flat 3D field space.
- Apply constraint $\phi^i \phi_i = r^2$
- Solutions form a sphere
- Can then replace $\phi^3 \longrightarrow (r^2 - (\phi^1)^2 - (\phi^2)^2)^{1/2}$
- Now only have two physical fields: ϕ^1, ϕ^2 - or θ^1 and θ^2

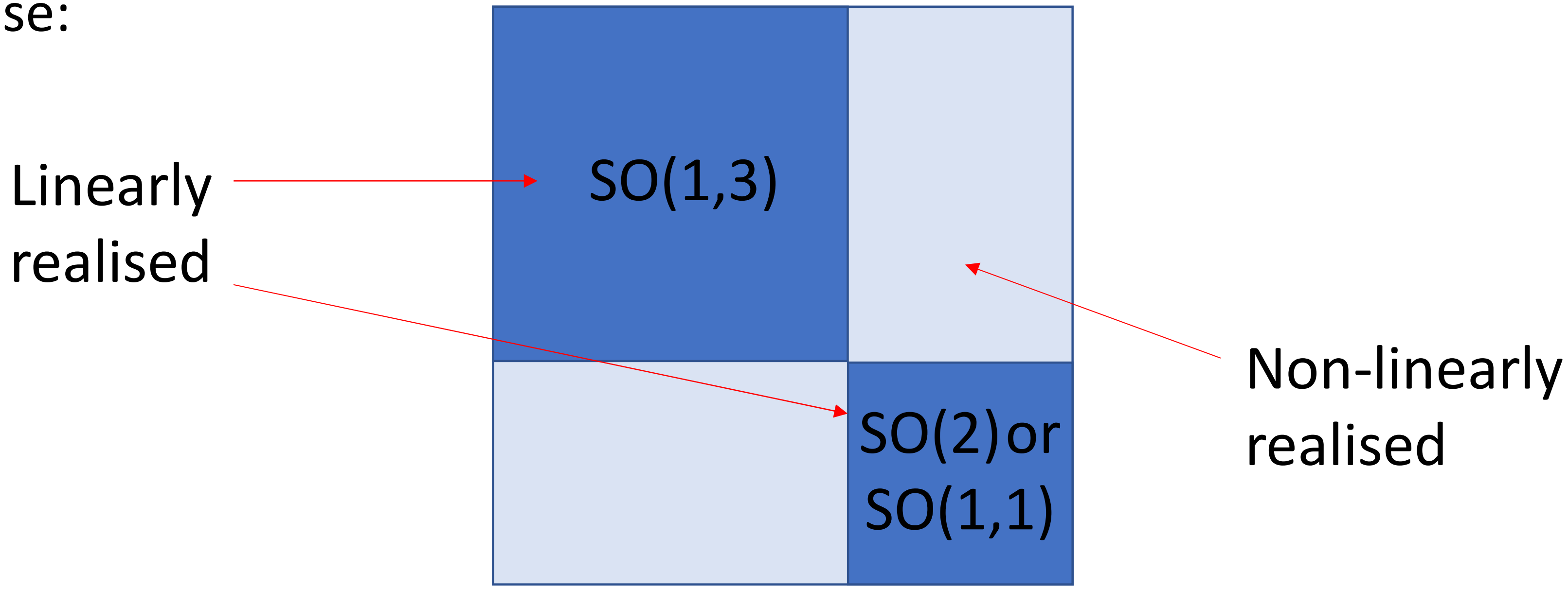
Constraint can result from a potential:

$$V = a^2(\phi^i \phi_i - r^2)^2$$

V is min (V = 0) for $\phi^i \phi_i = r^2$, so vacuum manifold is sphere.

My PhD

Ken was interested in groups SO(1,5) and SO(2,4). Non-linear realisation of these:

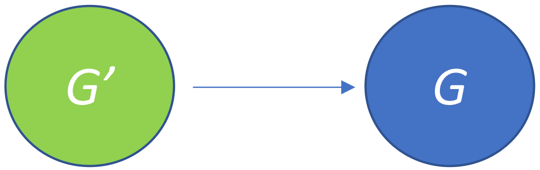


My role: to calculate key quantities for this non-linear realisation

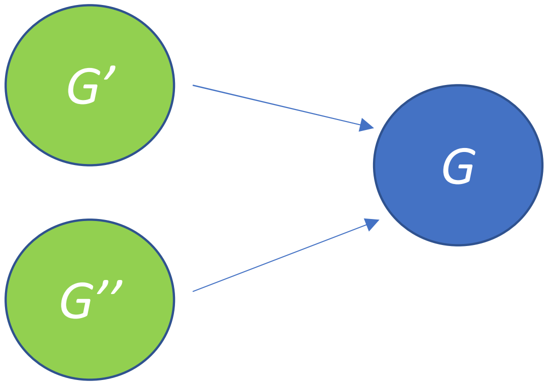
Spinor representations

- Ken: key to calculating these quantities is looking at spinor representations
- Definition: Representation of a group *G* is another group *G'* which has the same structure
- SO(*N*) groups:

N odd: $N = 2n + 1$



N even: $N = 2n$



Spinor representations

- Groups of 2^n x 2^n matrices, where $N = 2n$ or $N = 2n + 1$, e.g.
    - SO(3): 1 spinor rep; 2 x 2 matrices
    - SO(4): 2 spinor reps; 2 x 2 matrices
    - SO(5): 1 spinor rep; 4 x 4 matrices
    - SO(6): 2 spinor reps; 4 x 4 matrices
    - SO(7): 1 spinor rep; 8 x 8 matrices

SO(t,s) groups also have spinor reps,

- e.g. SO(1,3) has two 2-dimensional spinor reps – these describe spin ½ fields.

Covariant compactification – motivation and aim of programme

Higher-dimensional theories

My focus:

- Theories with >4 D, in which
  - The extra dimensions are space dimensions
  - They are curled up, or ‘compactified’ – far smaller than atomic scales
  - Applying GR concepts to this higher-dimensional spacetime
    $\longrightarrow$ gauge fields

Insight:

- *+2D, rotation matrices in these dimensions form an SO(2) group*
- *spinor rep is U(1) – gauge group of EM*
- *field equation could have SO(1,5) symmetry – but only gauge group and Lorentz group realised linearly*

Decompactification limit

- Consider a ‘decompactification limit’ – extra dimensions uncurl
- In this limit, all N dimensions appear on same footing (up to signature)
- Jacobian matrices form group GL(N, $\mathbb{R}$)

Using a multiplet to break GL(N, $\mathbb{R}$) symmetry

Key issue: choose fields for our equivalent of constraint $\phi^i \phi_i = r^2$

- Note that ϕ^i was (non-trivial multiplet) of SO(3)
- We therefore should not expect to use a scalar under coordinate changes to break GL(N, $\mathbb{R}$)
- GL(N, $\mathbb{R}$) has N^2 degrees of freedom; so does a rank-2 tensor $X_I{}^J$
- We want to trigger compactification – so choose tensor containing metric/connection/curvature tensor

Insight: the covariant derivative of a vector does all this:

$$\mathrm{D}_\mathrm{I} M^\mathrm{J} = \partial_\mathrm{I} M^\mathrm{J} + \Gamma^\mathrm{J}_{\mathrm{IK}} M^\mathrm{K}$$

Aim of covariant compactification programme

Aim – to find:

1. Constraints on $\mathrm{D_I}M^\mathrm{J}$ giving the correct symmetry breaking pattern: Lorentz, U(1), SU(2) and SU(3)…
2. A field equation/set of equations built from M^J, $\mathrm{D_I}M^\mathrm{J}$…
3. …such that:
   field eqn(s) + constraints $\longrightarrow$ GR + gauge theories (in appropriate limits)

Start by focusing on 1. – we can figure out a lot from it without needing field equations!

Covariant compactification – eigenvalues, product manifolds and gauge fields

Similarity transformations

Insight: under change of coordinates, action of $\mathrm{GL}(N, \mathbb{R})$ *on* $X_{\mathrm{I}}{}^{\mathrm{J}}$ *is*

$$X_{\mathrm{I}}{}^{\mathrm{J}} \mapsto j_{\mathrm{I}}{}^{\mathrm{K}} X_{\mathrm{K}}{}^{\mathrm{L}} \, (j^{-1})_{\mathrm{L}}{}^{\mathrm{J}}$$

- Preserves eigenvalues
- Multiplicities determine the 'breaking pattern' – which symmetries are realised linearly
- Eigenvalues are completely determined by traces of its powers:

$$\mathrm{tr}(X) = X_{\mathrm{I}}{}^{\mathrm{I}}; \quad \mathrm{tr}(X^2) = X_{\mathrm{I}}{}^{\mathrm{J}} X_{\mathrm{J}}{}^{\mathrm{I}}; \quad \mathrm{tr}(X^3) = X_{\mathrm{I}}{}^{\mathrm{J}} X_{\mathrm{J}}{}^{\mathrm{K}} X_{\mathrm{K}}{}^{\mathrm{I}}; \quad \ldots$$

Diagonalisable tensors

- X_{IJ} can be decomposed into symmetric + anti-symmetric parts. In mixed form:

$$X_{\mathrm{I}}^{\mathrm{J}} = X_{\mathrm{I}}{}^{\mathrm{J}} + X^{\mathrm{J}}{}_{\mathrm{I}}; \qquad X_{\mathrm{I}}{}^{\mathrm{J}} - X^{\mathrm{J}}{}_{\mathrm{I}}$$

- If tensor is diagonalisable, must be (mixed form of) symmetric tensor, $X_{\mathrm{I}}^{\mathrm{J}}$
- The presence of diagonalisable tensors tells us a lot about group theory + geometry of spacetime…

Example – Lorentz and SO(2)

…for example, if $X_{\mathrm{I}}^{\mathrm{J}}$ can be diagonalised to

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & b \end{pmatrix}$$

- it is invariant under GL(4, $\mathbb{R}$) $\otimes$ GL(2, $\mathbb{R}$)
  - these invariance groups are unbroken symmetries
  - valid in any coordinate system
  - they contain Lorentz group and an SO(2) group

Diagonalisable tensor fields and product manifolds

If any such tensor field $X_{\mathrm{I}}^{\mathrm{J}}$ can be diagonalised to this form across a region of spacetime, then

- the spacetime coincides with a product manifold across that region (see following slides)
- dimensionalities of its factor spaces = multiplicities of the eigenvalues

For proof, see T. Lawrence, *Product manifolds as realisations of general linear symmetries*, (2022), Int. J. Geom. Meth. Mod. Phys., doi:10.1142/S0219887822400060

Note: a and b don't need the same values everywhere – just the same multiplicities (thus scalar fields)

Product manifolds - definition

- Spaces with a block diagonal metric, in appropriate coordinates:

$$g_{\mathrm{IJ}} = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & g_{\psi\chi} \end{pmatrix}$$

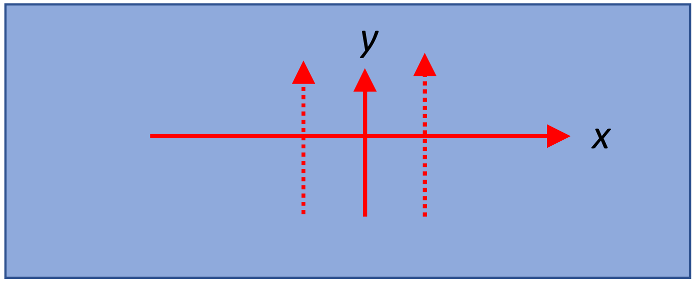
- Not unusual – most spacetimes of interest in GR are products of 4 1D spaces:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{pmatrix}$$

Product manifolds – simple examples

2D flat plane

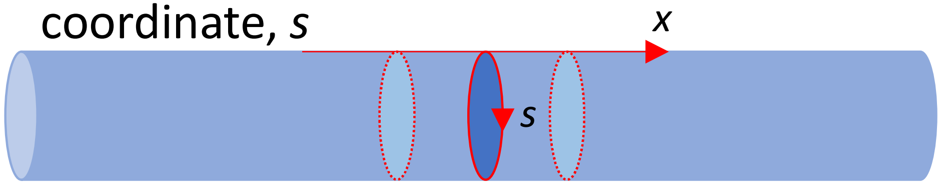
- Draw a straight line through it; take distance along this line as a coordinate *x*
- Straight line cross-section at each value of *x*; distance along this line is the second coordinate *y*



- In this sense, plane is a product of a two straight lines
- These lines are its factor spaces

2D cylinder

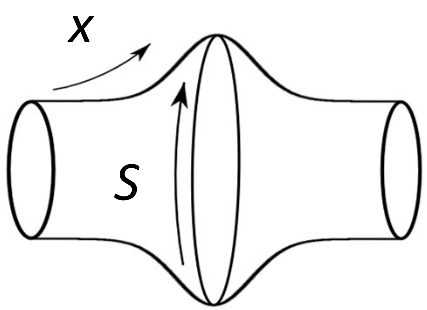
- Draw a line along the length of the cylinder; take distance along this line as a coordinate, *x*
- At each value of this coordinate, there is a circular cross-section. Distance moved around this circle is the second coordinate, *s*



- Cylinder is therefore a product of a line and a circle
- The line and the circle are its factor spaces

Product manifolds – example: tube

Tube of varying radius



- This still has a line along its length, which can be used to define one coordinate, *x*
- Still has a circular cross-section at each point on this line, so can be used to define second coordinate, *s*
- But now the size of the circle varies along the line

Product manifolds and Kaluza-Klein theories

Given how common product manifolds are, my result may not seem too impressive

But it's important for Kaluza-Klein:-

If $X_{\mathrm{I}}^{\mathrm{J}}$ can be diagonalised across a region to

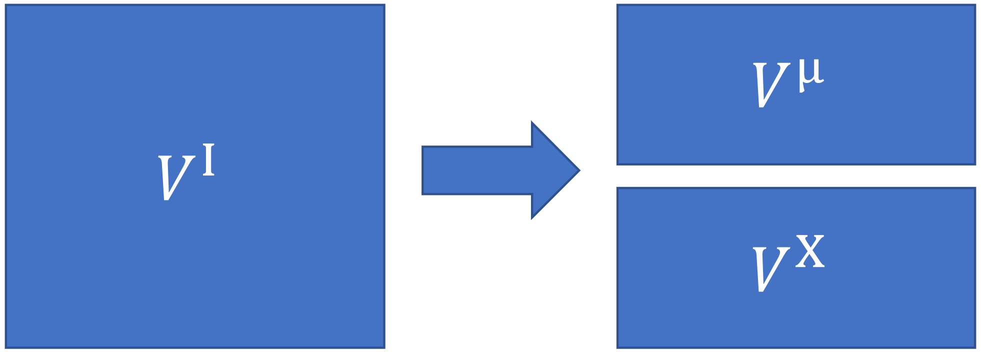
$$\begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & b \end{pmatrix}$$

Then

- the higher-dimensional spacetime coincides with a product space across a region
- one of the factor spaces is our usual four-dimensional spacetime

Product manifolds and Kaluza-Klein theories

On such manifolds, we can decompose tensors in terms of the factor spaces, e.g.



The ‘cylinder condition’

Similarly, the Levi-Civita connection components can be assembled into subsets:

$$\Gamma_{\mathrm{JK}}{}^{\mathrm{I}} = \{\Gamma_{\nu\kappa}{}^{\mu}, \Gamma_{\nu X}{}^{\mathrm{Y}}, \Gamma_{\mathrm{XY}}{}^{\mu}, \Gamma_{\nu\mathrm{Y}}{}^{\mu}, \Gamma_{\nu\kappa}{}^{\mathrm{Y}}, \Gamma_{\mathrm{XY}}{}^{\mathrm{Z}}\}$$

- Several of these vanish if $g_{\mu\nu}$ is independent of the y^{X} coordinates
- This means cross-section of tube-like spacetime is S^1, S^2, S^3, ...

Gauge fields as connection components

If we use:

- y^{X} coordinates on the 4D spacetime
- coordinates with orthonormal basis on the other factor space

then $\Gamma_{\nu X}{}^{\mathrm{Y}}$ become gauge fields of SO(N) – or equivalently, corresponding unitary symmetry, e.g.

- for +3D, with appropriate assignments,

$$\mathrm{D}_{\mathrm{\mu}} V^{Y} = \partial_{\mathrm{\mu}} V^{Y} - igA_{\mathrm{\mu}}^{\mathrm{i}} (\mathrm{T_i})_X^Y V^X$$

- for +2D

$$D_{\mathrm{\mu}} V = \partial_{\mu} V + iqA_{\mathrm{\mu}} V; \quad D_{\mathrm{\mu}} V^* = \partial_{\mu} V^* - iqA_{\mathrm{\mu}} V^*$$

Field strength as Riemann tensor components

- $F_{\mathrm{\mu\nu X}}{}^{\mathrm{Y}}$ can be found in Riemann tensor components for these coordinates

- It doesn't contribute to $R_{\mathrm{\mu\nu}}$ or R_{XY}

- We can view gauge fields as variations in "radius" along "tube"

Summary so far

Construct gauge fields as follows:

- Define M^{J} and $\mathrm{D_I}M^{\mathrm{J}}$
- $\mathrm{D_I^J} = \mathrm{D_I}M^{\mathrm{J}} + \mathrm{D_J}M^{\mathrm{I}}$ forms 'orbits' under changes of coordinates
- Orbits containing diagonal matrices characterised by $\mathrm{D_I^I}$, $\mathrm{D_I^J D_J^I}$, $\mathrm{D_I^J D_J^K D_K^I}$,…
- If these result in same multiplicities of eigenvalues over region:
    - Spacetime coincides with product space
    - Invariants determine dimensionalities of factor spaces
    - Each factor space has its own curvature.
    - Unitary gauge fields can be found in connection components; field strength in Riemann tensor components

Points to note

Note that:

- We have not yet used the antisymmetric part: $\mathrm{D_I} M^{\mathrm{J}} - \mathrm{D_J} M^{\mathrm{I}}$
- We have not yet used any field equations

Covariant compactification – (first) field equation

Field equation and scalar invariants

- Field equation $\longrightarrow$ curvature and matter distribution. Derive by
  - constructing Lagrangian (density) from invariants
  - applying Euler-Lagrange equation
- For $\mathrm{D_I}M^\mathrm{J}$, invariants are again traces of powers; 2nd looks like a kinetic term:

$$\mathrm{D_I}M^\mathrm{J}\,\mathrm{D_J}M^\mathrm{I}$$

- But want extra dimensions to be tightly compact – need a mass term:

$$M_\mathrm{I}M^\mathrm{I}$$

- Therefore try

$$\mathcal{L} = \mathrm{D_I}M^\mathrm{J}\,\mathrm{D_J}M^\mathrm{I} - kM_\mathrm{I}M^\mathrm{I}$$

where k is a constant (dimensionful, but invariant and constant across spacetime)

Lagrangian and Euler-Lagrange equation

- Principle of Least Action $\longrightarrow$ Euler-Lagrange equation:

$$\mathrm{D_I D_J} M^{\mathrm{I}} = -\, k\, M_{\mathrm{J}}$$

- Can also be derived as simplest generalisation of Poisson equation for gravity for a vector field which is:
  - consistent with general covariance
  - consistent with equivalence principle
- This method provides a value for k:

$$k = \frac{4\pi G\rho}{c^2}$$

where ρ is vector field's density

Ricci form

By using the relation

$$[\mathrm{D_K}, \mathrm{D_J}]M^{\mathrm{I}} = \mathrm{R^I}_{\mathrm{LKJ}}M^{\mathrm{L}}$$

we get the following form for the E-L equation:

$$(R^{\mathrm{I}}_{\mathrm{J}} + \mathrm{D^I D_J})M^{\mathrm{J}} = -\frac{4\pi G\rho}{c^2}M^{\mathrm{I}}$$

Insight: this is an eigenvalue equation for M^{I}

An operator equation incorporating geometry

Compare and contrast it with key equations in QM, RQM and GR:

- Like Schrödinger, Dirac and Klein-Gordon, it has a second-order differential operator
- But those three assume (pseudo)-orthonormal coordinates on:
  - A flat 3D space (Schrödinger)
  - A flat 4D spacetime (Dirac and Klein-Gordon)
- By contrast, this incorporates geometry into the operator
- Like Einstein field equation, it relates geometry to matter

Solutions – starting observations

Further work needs to be done on solutions – *but this may not be final field equation of theory (see later)*

We *can* say:

- Unlike solving Schrödinger eqn as undergrad, vector M^{L} is not only unknown – also $\Gamma_{\mathrm{JK}}{}^{\mathrm{I}}$ is unknown in operator
- Solutions would give relation between geometry and M^{L}
- Doesn’t tell us about action of operator on other vectors

Solutions – early thoughts

- There are values of $R^{\mathrm{I}}_{\ \mathrm{J}}$ for which *every* vector is an eigenvector – when

$$R^{\mathrm{I}}_{\ \mathrm{J}} \propto \delta^{\mathrm{I}}_{\ \mathrm{J}}$$

  - Einstein manifolds (for D>2)

- May be similar cases for $R^{\mathrm{I}}_{\ \mathrm{J}} + \mathrm{D}^{\mathrm{I}}\mathrm{D}_{\mathrm{J}}$
- Also solutions for specific M^{L}
- For

$$\mathrm{D}_{\mathrm{J}} M^{\mathrm{I}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b(y^{\mathrm{X}}) & 0 \\ 0 & 0 & 0 & 0 & 0 & b(y^{\mathrm{X}}) \end{pmatrix}$$

  product of Minkowski spacetime and two-sphere is solution
  – vacuum manifold of 6D Kaluza-Klein theory

What we've done and what's missing

What we've done (so far)		What's missing (so far)	
Field equation which contains geometric d.o.f.	✓	SU(3) gauge fields	✗
Constraints → Product spacetime	✓	Quarks, leptons	✗
4D gravity	✓	Electroweak symmetry breaking (Higgs?)	✗
SU(2) and U(1) gauge fields	✓	Quantum numbers	✗
Solution representing classical vacuum	✓	Field equation for gauge fields	✗

Covariant compactification – spinors and SU(3)

Spinors

- Include spinor reps in the model using Dirac's kets & bras notation
- If ψ^{α} transforms as *d*-dimensional spinor rep of SO(*N*), $\psi^{\alpha} \in K^{d}$
- Ket space K^{d} has inner product, $< \boldsymbol{\psi} | \boldsymbol{\chi} >$
- Inner product defines a 'spinor metric':

$$< \boldsymbol{\psi} | \boldsymbol{\chi} >= \xi_{\alpha\beta}\psi^{\alpha}\chi^{\beta}$$

- Not symmetric – instead satisfies

$$(\xi_{\beta\alpha})^{*} = \xi_{\alpha\beta}$$

Transformations on spinors

Ket spaces can then be analysed in a similar way to tangent spaces

Tangent space	**Ket space**
Vector has *N* real components V^{J} - relating to particular basis on tangent space	Ket has *d* complex components ψ^{α} - relating to particular basis on ket space
Basis transformed by real invertible *N* x *N* matrix $j_{\mu}{}^{\kappa} \in \mathrm{GL}(4, \mathbb{R})$	Basis is transformed by complex invertible *d* x *d* matrix $z_{\alpha}{}^{\beta} \in \mathrm{GL}(4, \mathbb{C})$
(Pseudo-)orthonormal basis on tangent space preserved by (pseudo-)orthogonal matrices $i_{\mu}{}^{\kappa}$	Orthonormal basis on ket space preserved by unitary matrices $u_{\alpha}{}^{\beta}$
One tangent space at each point in space(time). Use “bundle” of tangent spaces to construct vector fields	Attach ket space to each point in spacetime. Use “bundle” of ket spaces to construct spinor fields

Spinors as complex tangent vectors?

- Can postulate complex, curved *d*-dimensional manifold, C^{d}
- Then $K_P^d = \mathrm{T}_P C^d$
- Then $z_{\alpha}{}^{\beta} \in$ GL(4, $\mathbb{C}$) represent change of coordinates
- Not necessary for most of the analysis…
- …but may help for spinor field equation – see later

Outer products

- To each ket, there corresponds a bra, with components

$$\psi_\alpha = \xi_{\alpha\beta}\psi^\beta$$

- Create outer product $\chi_\alpha\psi^\beta$ - has d^2 components
- This matrix can be written as complex linear sum of generators of SU(d) and **1** - e.g. for SU(2)

$$\chi_\alpha\psi^\beta = \theta^1(T_1)_\alpha{}^\beta + \theta^2(T_2)_\alpha{}^\beta + \theta^3(T_3)_\alpha{}^\beta + \theta^4(\mathbf{1})_\alpha{}^\beta$$

- Transforms by conjugation under $z_\alpha{}^\beta$ (and hence $u_\alpha{}^\beta$):

$$\chi_\alpha\psi^\beta \mapsto z_\alpha{}^\beta\chi_\alpha\psi^\beta\,(z^{-1})_\alpha{}^\beta$$

Diagonalising the outer product

- When all θs are real, $\chi_\alpha\psi^\beta$ is Hermitian
- Hermitian matrix may always be diagonalised by an appropriate unitary one
- Again, diagonal form tells us about the group theory – e.g. if $\chi_\alpha\psi^\beta$can be diagonalised to

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$$

then it is invariant under SU(3) $\otimes$ U(1)

SU(4) and SO(6)

For 4-component spinors, alternative decomposition is useful

Generators of SU(4) $\longleftrightarrow$ Generators of SO(6)

For details, see K J Barnes, J Hamilton-Charlton and T R Lawrence, *How orbits of SU(N) can describe rotations in SO(6)*, (2001), J. Phys. A 34, 10881

Thus $\chi_\alpha \psi^\beta$ can also be expressed as:

$$\chi_\alpha \psi^\beta = \omega^{12}(T_{12})_\alpha{}^\beta + \omega^{13}(T_{13})_\alpha{}^\beta + \cdots + \omega^{56}(T_{56})_\alpha{}^\beta + \theta(\mathbf{1})_\alpha{}^\beta$$

where ω^{XY} and T_{XY} are all (complex) anti-symmetric matrices

From SO(6) parameters to SU(3) symmetry

- Eigenvalues are then determined by

$$\omega^{12}, \omega^{13}, \ldots \omega^{56}, \theta$$

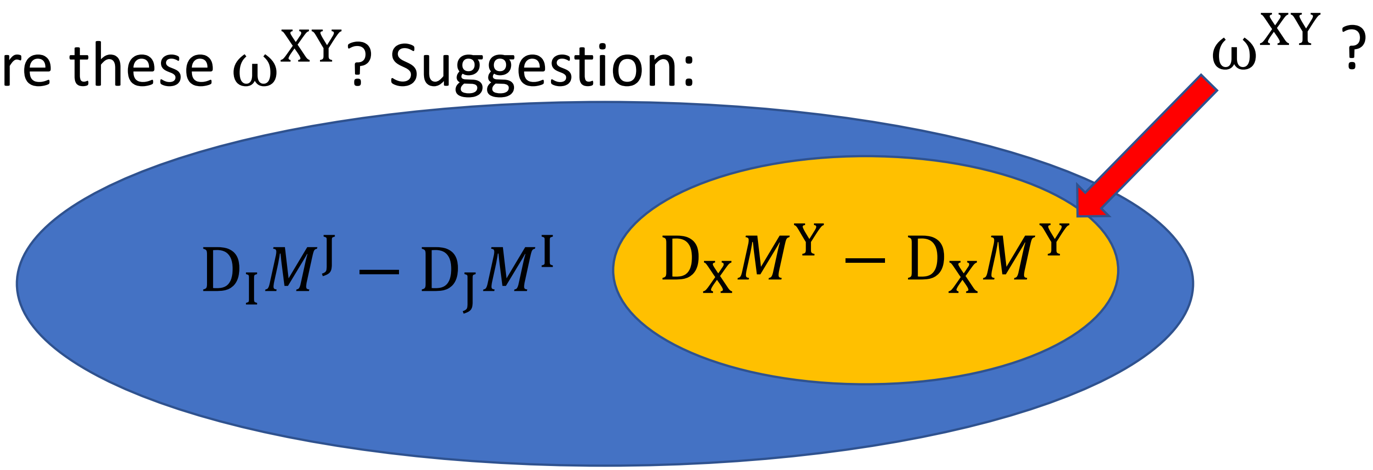
- If eigenvalues are real, $\chi_\alpha \psi^\beta$ can be diagonalised using SU(4)
- Can therefore choose $\omega^{12}, \omega^{13}, \ldots \omega^{56}, \theta$ such that $\chi_\alpha \psi^\beta$ is invariant under SU(3) $\otimes$ U(1)
- Again, these will be unbroken symmetries
- This is how we get colour SU(3) – the gauge group of the strong force – into the theory

Meaning of ω^{XY}

In fact, we don't need θ – can obtain $\chi_\alpha \psi^\beta$ with desired diagonal form just using ω^{XY}

(This is not to say θ won't play a role in the theory – that remains to be seen!)

So what are these ω^{XY}? Suggestion:



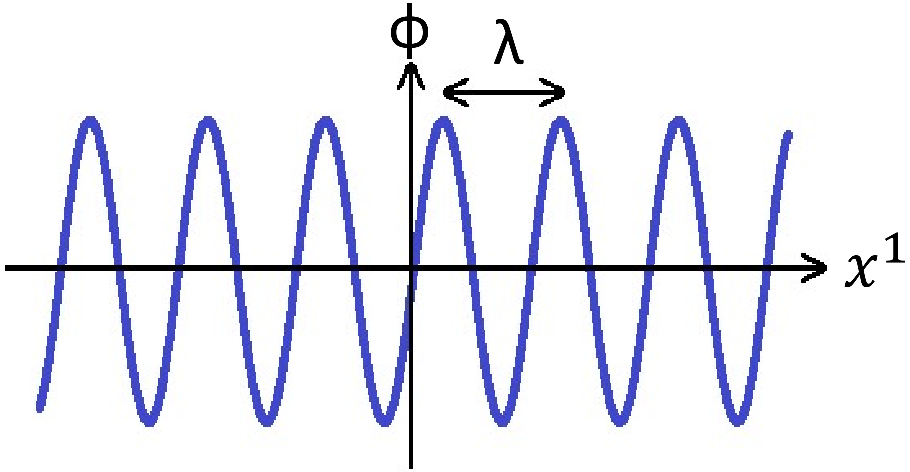
Covariant compactification – linear coordinate transformations and field configurations

Linear transformations

- Consider decompactification limit: flat N-dimensional spacetime
- Minkowski coordinates x^I : $\eta^{(x)}_{IJ} = diag(1, -1, -1, -1)$
- Linear coordinate transformation: $x'^I = z^I{}_J x^J + \epsilon^I$
    - $z^I{}_J \epsilon$ GL(N, $\mathbb{R}$)
    - Tangent space is invariant under ϵ^I

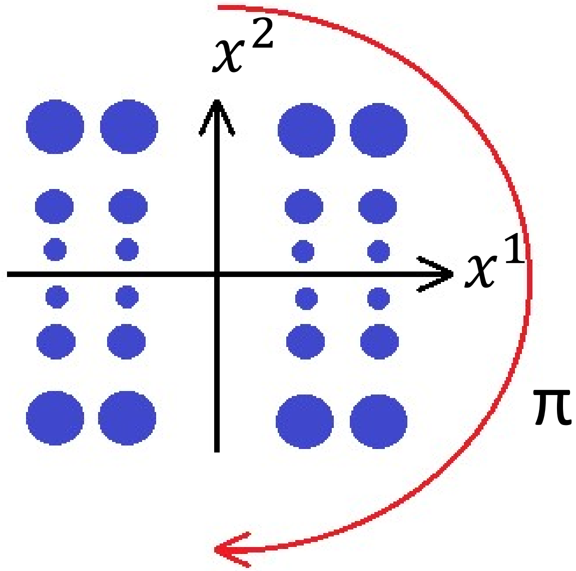
Symmetric field configurations - periodicity

Translation symmetry:



– symmetric under translation through λ

Rotation symmetry:



– symmetric under rotation through π

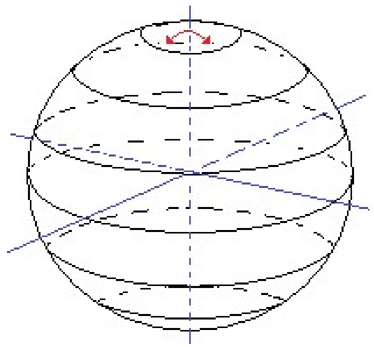
Operators, eigenfunctions, eigenvalues

- Generated by differential operators, e.g.
  - Translations generated by $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}, \ldots$
  - 2D rotations generated by $\frac{\partial}{\partial \theta}$
- Eigenfunctions form basis:
  - Fourier modes for translations
  - Harmonics for rotations
- Eigenvalues inversely proportional to the period. These are the quantum numbers.

Translations and compactification

- On compactification, $x^I \longrightarrow u^I$
- Displacements: $u'^I = u^I + \epsilon^I$
- Extra dimensions:

- For spherical space, these are rotations



Compactification, rotations and harmonics

- Displacements don't commute with each other, or with gauge transformations – e.g. for 2D, $SO(3) \supset SO(2)$
- Field configurations: no longer any value of λ
- Instead, must be harmonics

(Theory then exploits loophole in O'Raifeartaigh's theorem, as discussed briefly in T. Lawrence, *Product manifolds as realisations of general linear symmetries*, (2022), Int. J. Geom. Meth. Mod. Phys., doi:10.1142/S0219887822400060)

Covariant compactification – developing field equations

Bianchi identities

- So far, we have first stab at field equation for M^{J} :

$$\mathrm{D_I D^J} M^{\mathrm{I}} = \left(R^{\mathrm{I}}_{\mathrm{J}} + \mathrm{D^I D_J}\right) M^{\mathrm{J}} = -\frac{4\pi G\rho}{c^2} M^{\mathrm{J}}$$

- We have compared with GR field equation
- But in GR, there is another constraint on geometry – Bianchi identity:

$$D_\mu R_{\nu\rho\sigma\kappa} + D_\kappa R_{\nu\rho\mu\sigma} + D_\sigma R_{\nu\rho\kappa\mu} = 0$$

- It has a contracted form:

$$D_\mu G^\mu_\nu = 0$$

The full identity and the gauge field equation

- These are geometric facts – independent of geometry-matter relation
- Have implications for covariant compactification
- From the full Bianchi identity, we find

$$D^{\nu} F_{\mu\nu X}{}^{Y} = D^{Y} G_{\mu X} - D_{X} G_{\mu}{}^{Y}$$

- RHS carries gauge field indices; transforms as Lorentz vector
- Perhaps: this eqn + field eqn for M^J + constraints on $D_I M^J$

$$\longrightarrow D^{\nu} F_{\mu\nu} = g j_{\mu} \, ?$$

The contracted identity & Noether’s theorem

- Now turn to contracted Bianchi identity
- Noether developed a theorem for conservation in GR
- Starts with

$$S = \int (-|g_{\mu\nu}|)^{1/2} R \, \mathrm{d}\Omega$$

- Carries out change of coordinates, which vanishes on a boundary
- Result:

$$D_\mu G^\mu_\nu = 0$$

– the contracted Bianchi identity

Questioning the principle of least action

- Now see: contracted identity = "conserved geometry"
- Also leads us to question:

  Are action & principle of least action really fundamental?

- Why should action be extremised – under variation which vanishes on surface?

Field equation without least action?

Question: could we derive a field equation without using least action?

Answer: I think so – but does it have a “real world” solution?

Basic principles:

- $M^I M_J$ is scalar; under displacement (parallel transport) $\delta(M^I M_J)$ is also scalar
- Same goes for $\mathrm{D_I}M^\mathrm{J}\ \mathrm{D_J}M^\mathrm{I}$ and $\delta(\mathrm{D_I}M^\mathrm{J}\ \mathrm{D_J}M^\mathrm{I})$
- We can then define a scalar field ρ as a ratio between variations
- Then

$$\delta(\mathrm{D_I}M^\mathrm{J}\ \mathrm{D_J}M^\mathrm{I}) - \rho\, \delta(M^I M_J) = 0$$

→ field equation. But solutions require better understanding of $\mathrm{D_I}M^\mathrm{J} - \mathrm{D_J}M^\mathrm{I}$

Spinor field equation

- Can now speculate: what might spinor field equation look like
- Say $\hat{O}M = OM$
- Then look for $\hat{o}|\boldsymbol{\psi}>$ = $o|\boldsymbol{\psi}>$

  such that $\hat{O}M = OM$ where $M^I = <\boldsymbol{\psi}|\gamma^I|\boldsymbol{\psi}>$
- "square root" of field equation for M^I
- Eigenvalues of $\hat{O}$ $\longrightarrow$ spacetime geometry
- ? Eigenvalues of $\hat{o}$ $\longrightarrow$ geometry of curved complex space ?

Quantum numbers of the Standard Model

Counting fermion states

- Covariant compactification $\Rightarrow$ "GraviGUT" for fermions
- 1^{st} generation – count states:
  - Quarks: 2 (spin) x 2 (chirality) x 3 (colour) x 2 (weak isospin) = **24**
  - Charged leptons: 2 (spin) x 2 (chirality) x 1 (weak isospin) = **4**
  - Neutrinos: 2 (spin) x 2 (chirality) x 1 (weak isospin) = **4**
- **TOTAL = 32**
- I have found explicit 32 x 32 forms for generators of

$$\mathrm{Spin}\,(1,3) \otimes \mathrm{SU}(3)_C \otimes \mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$$

How many dimensions does spacetime have?

- 2 families of SO groups have 32-component spinor:
  - 32: SO(11), SO(1,10), SO(2,9), SO(3,8)...
  - $16 \oplus 16$: SO(10), SO(1,9), SO(2,8), SO(3,7)...
- We need these to contain SO(1,3) and SO(6) ~ SU(4) $\supset$ SU(3) $\otimes$ U(1)
- Thus two possibilities with one time dimension: SO(1,10), SO(1,9)
- Implies spacetime has 10 or 11 dimensions $\longrightarrow$ SUGRA, strings?

Next steps and outstanding questions

- Next steps:

1. express generators of

$$\mathrm{Spin}\ (1,3) \otimes \mathrm{SU}(3)_C \otimes \mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$$

in terms of Clifford algebra

2. Find induced action on vectors

- Big remaining questions:
  - Fermion generations?
  - Higgs?

Questions?