# Geometry, gravity and spin

## Notes, ideas and developments

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# Motivating questions

# Outline

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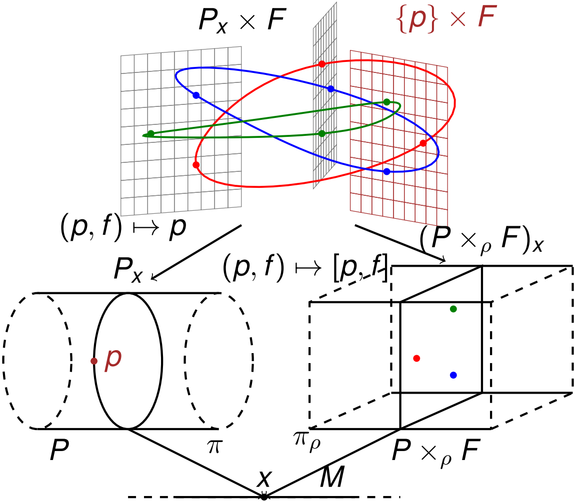
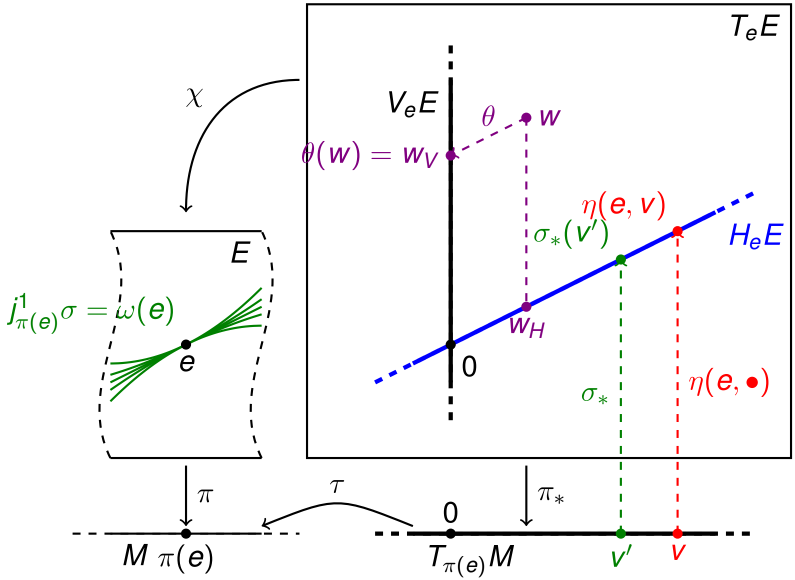
1. Bundles, connections, Cartan geometry
2. Unification of field theories from particle motion
3. Spinors and spontaneous symmetry breaking in gravity
4. Observers, Ashtekar variables and loop quantization

# The associated bundle



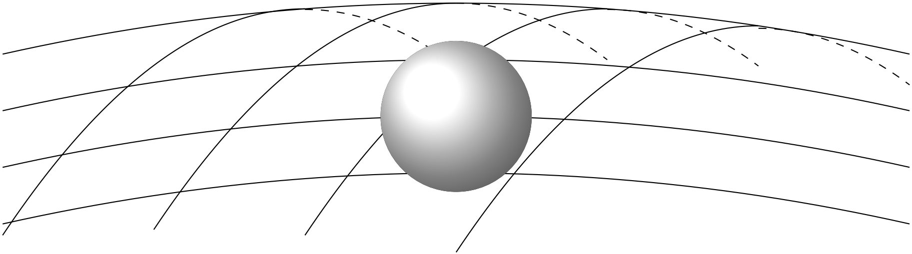
# The many faces of connections



# A hamster's perspective on Cartan geomeetry

- Cartan geometry: view from inside a Hamsster ball.
- All possible motions of the hamster: group $G$.
- Motions which will not move the ball: subgroup $H \subset G$.
- Cartan connection $A$ connects motions of hamster and ball.
- Cartan curvature: difference between ball and surface geometry.

$$F = dA + \frac{1}{2}[A, A]\,.$$



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# Charged particle in gauge theory

- Ingredients of the theory:
  - Lie group $K$ with Lie algebra $\mathfrak{k}$.
  - Lie algebra generators (basis) $T_A$ and structure constants $f_{AB}{}^C$:

$$[T_A, T_B] = f_{AB}{}^C T_C \,. \tag{1}$$

  - Bilinear form $\beta_{AB} = \beta(T_A, T_B)$ .
  - Representation $\rho$ of $G$ inducing linear map $z^a \mapsto \rho^a{}_{bA} X^A z^b$.

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  - Representation $\rho$ of $G$ inducing linear map $z^a \mapsto \rho^a{}_{bA} X^A z^b$.
- Dynamical variables in the theory:
  - Gauge field $A^A_\mu$ with field strength $F = \mathrm{D}A$:

$$F^A{}_{\mu\nu} = \partial_\mu A^A{}_\nu - \partial_\nu A^A{}_\mu + A^B{}_\mu A^C{}_\nu f_{BC}{}^A\,. \tag{2}$$

  - Particle at $x^\mu$, velocity $y^\mu = \dot{x}^\mu$, isospin $z^a$.

# Particle motion in field theories

1. Geodesic motion in general relativity:
   - Geodesic equation depends on connection coefficients $\Gamma^{\mu}{}_{\nu\rho}$:

$$\ddot{x}^{\mu} + \Gamma^{\mu}{}_{\nu\rho}\dot{x}^{\nu}\dot{x}^{\rho} = 0\,. \qquad (3)$$

   - First order formulation with velocity coordinate $y^{\mu} = \dot{x}^{\mu}$:

$$\dot{x}^{\mu} = y^{\mu}\,, \quad \dot{y}^{\mu} = -\Gamma^{\mu}{}_{\nu\rho}y^{\nu}y^{\rho}\,. \qquad (4)$$

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   - Internal degree of freedom:

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⇝ Formal similarities - common geometric description?

# Unified ansatz in Cartan geometry

1. Geodesic motion in general relativity:
   - Spacetime manifold $M$ with Lorentzian metric $g$.
   - Orthonormal frame bundle $P_0$ is principal $H_0$-bundle, $H_0 = \mathrm{SO}(1,3)$.
   - Tangent bundle $TM = P_0$ is associated to $P_0$.
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2. Include internal degrees of freedom:
   - Enlarge model geometry to $G/H$ with $G = G_0 \times K$ and $H = H_0 \times K$.
   - Cartan connection $A$ is $\mathfrak{g}$-valued 1-form on $H$-bundle $P$.
   - $A$ splits into solder form, Levi-Civita & gauge connections.
   - Associated bundle $E = TM \times_M F$ includes velocity and isospin.

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⇝ Combined description:
   - Field theory combines Palatini and Yang-Mills Lagrangians.
   - Particle equation of motion from Lagrangian with gauge coupling.
   - Particle trajectories as Integral curves of combined vector field.

# Aside: time translation in Finsler-Cartan geometry

- Consider the fundamental vector field

$$\mathbf{t} = \underline{A}(\mathcal{Z}_0) = f_0^a \partial_a - f_j^a N^b{}_a \bar{\partial}_b^j \qquad \Leftrightarrow \qquad \omega^i{}_j(\mathbf{t}) = 0\,, \quad e^i(\mathbf{t}) = \delta_0^i\,. \tag{7}$$

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- From $\omega^\alpha{}_\beta(\mathbf{t}) = 0$ follows:

$$0 = \dot{f}_\alpha^a + f_\alpha^b\left(\dot{x}^c F^a{}_{bc} + (\dot{x}^d N^c{}_d + \dot{f}_0^c)C^a{}_{bc}\right) = \nabla_{(\dot{x},\dot{f}_0)}f_\alpha^a\,. \tag{10}$$

$\Rightarrow$ Frame $f$ is parallely transported.

# Outline

# From Stelle-West to MacDowell-Mansouri

- Stelle-West action uses $\mathfrak{g}$-valued curvature form $F^{AB}$:

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- Idea: symmetry breaking field $y^A = \bar{\Psi}\Gamma^A\Psi$ with higher spinor $\Psi$?

# Dirac-Cartan-Higgs action and symmetry breaking

- Ingredients:
  - Spin groups $\bar{G}$ and $\bar{H}$: double covers of $G$ and $H$.
  - Spin frame bundle $\bar{P}$ is principal $\bar{H}$ bundle.
  - Right action of $\bar{H}$ on $\bar{G}$ yields principal $\bar{G}$-bundle $\bar{Q} = \bar{P} \times_{\bar{H}} \bar{G}$.
  - Levi-Civita connection gives Ehresmann connection $\tilde{A}^{AB}$ on $\bar{Q}$.
  - "Higher" spinor $\Psi$: section of bundle associated to $\bar{Q}$.
  - Dirac matrices $\Gamma^A$ acting on $\Psi$.

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  - "Higher" spinor $\Psi$: section of bundle associated to $\bar{Q}$.
  - Dirac matrices $\Gamma^A$ acting on $\Psi$.
- Construction of Dirac-Cartan-Higgs action:
  - Exterior covariant derivative of higher spinor:

$$\mathrm{D}\Psi = \mathrm{d}\Psi + \frac{1}{8}\tilde{A}^{AB} \wedge [\Gamma_A, \Gamma_B]\Psi \,. \tag{13}$$

  - Dirac-Cartan operator:

$$\not{D}\Psi = \rho([\mathbb{E}^A, \mathbb{E}^B])\underline{\tilde{A}}_{AB} \lrcorner \mathrm{D}\Psi \,. \tag{14}$$

  - Dirac-Cartan-Higgs action:

$$\mathcal{L} = \bar{\Psi}\not{D}\Psi + m\bar{\Psi}\Psi - \lambda(\bar{\Psi}\Psi)^2 \,. \tag{15}$$

# Outline

# Sketch

- Lorentz group $H$ contains rotation subgroup $K \subset H$.
- Principal bundle $P$ of frames reduces to $K$-bundle.
- Base space: “observer space” $O$ of normalized velocities.
- Transition to spin groups $\bar{G}$, $\bar{H}$, $\bar{K}$.
- Cartan connection contains $\mathfrak{k}$-valued part.

$\Rightarrow$ Ashtekar variables.