Geometry, gravity and spin
Notes, ideas and developments

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Outline

1. Bundles, connections, Cartan geometry
2. Unification of field theories from particle motion
3. Spinors and spontaneous symmetry breaking in gravity
4. Observers, Ashtekar variables and loop quantization
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The associated bundle

\[ P_x \times F \quad \{p\} \times F \]

\[ (p, f) \mapsto p \]

\[ (P \times \rho F)_x \]

\[ (p, f) \mapsto [p, f] \]

\[ P \]

\[ P \times \rho F \]

\[ \pi \]

\[ \pi_\rho \]

\[ M \]
The many faces of connections

\[ j^1_{\pi(e)} \sigma = \omega(e) \]

\[ \theta(w) = w_v \]

\[ \eta(e, v) \]

\[ \sigma^*(v') \]

\[ \pi(e) \]

\[ \pi_*(e) \]

\[ \tau \]

\[ \chi \]
A hamster’s perspective on Cartan geometry

- Cartan geometry: view from inside a hamster ball.
- All possible motions of the hamster: group $G$.
- Motions which will not move the ball: subgroup $H \subset G$.
- Cartan connection $A$ connects motions of hamster and ball.
- Cartan curvature: difference between ball and surface geometry.

\[ F = dA + \frac{1}{2}[A, A]. \]
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Charged particle in gauge theory

Ingredients of the theory:
- Lie group $K$ with Lie algebra $\mathfrak{k}$.
- Lie algebra generators (basis) $T_A$ and structure constants $f_{AB}^C$:
  \[ [T_A, T_B] = f_{AB}^C T_C. \] (1)
- Bilinear form $\beta_{AB} = \beta(T_A, T_B)$.
- Representation $\rho$ of $G$ inducing linear map $z^a \mapsto \rho_{ba}^a X^A z^b$.
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- **Dynamical variables in the theory:**
  - Gauge field $A^A_\mu$ with field strength $F = DA$:
    \[
    F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + A^B_\mu A^C_\nu f_{BC}^A .
    \] (2)
  - Particle at $x^\mu$, velocity $v^\mu = \dot{x}^\mu$, isospin $z^a$. 

1. Geodesic motion in general relativity:
   ○ Geodesic equation depends on connection coefficients $\Gamma^\mu_{\nu\rho}$:
     \[ \ddot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho = 0. \] (3)
   ○ First order formulation with velocity coordinate $y^\mu = \dot{x}^\mu$:
     \[ \dot{x}^\mu = y^\mu, \quad \dot{y}^\mu = -\Gamma^\mu_{\nu\rho} y^\nu y^\rho. \] (4)

2. Charged particle in Yang-Mills (non-abelian gauge) theory:
   ○ Internal degree of freedom:
     \[ \dot{z}^a = -\rho_a^b A^b A^\mu z^b y^\mu. \] (5)
   ○ Force equation depending on curvature $F = D A^? \dot{y}^\mu$.
Particle motion in field theories

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\[\Rightarrow\] Formal similarities - common geometric description?
1. Geodesic motion in general relativity:
   - Spacetime manifold $M$ with Lorentzian metric $g$.
   - Orthonormal frame bundle $P_0$ is principal $H_0$-bundle, $H_0 = \text{SO}(1,3)$.
   - Tangent bundle $TM = P_0$ is associated to $P_0$.
   - $\Gamma^{\mu\nu\rho}$ induces Cartan connection on $P$.
   $\Rightarrow$ Cartan geometry with model $G_0/H_0$ on $P$. 

2. Include internal degrees of freedom:
   - Enlarge model geometry to $G/H$ with $G = G_0 \times K$ and $H = H_0 \times K$.
   - Cartan connection $A$ is $g$-valued 1-form on $H$-bundle $P$.
   - $A$ splits into solder form, Levi-Civita & gauge connections.
   - Associated bundle $E = TM \times M$ includes velocity and isospin.
   $\Rightarrow$ Combined description:
   - Field theory combines Palatini and Yang-Mills Lagrangians.
   - Particle equation of motion from Lagrangian with gauge coupling.
   - Particle trajectories as Integral curves of combined vector field.
Unified ansatz in Cartan geometry

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Aside: time translation in Finsler-Cartan geometry

- Consider the fundamental vector field

  \[ t = A(Z_0) = f_0^a \partial_a - f_j^a N^b a \bar{\partial}_b \quad \Leftrightarrow \quad \omega^{i \ j}(t) = 0, \quad e^i(t) = \delta^i_0. \quad (7) \]

- Integral curve \( \Gamma : \mathbb{R} \to P, \lambda \mapsto (x(\lambda), f(\lambda)) \) of \( t \).

- From \( e^i(t) = \delta^i_0 \) follows:

  \[ \dot{x}^a = f_a^0. \quad (8) \]

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- From \( \omega^{\alpha \beta}(t) = 0 \) follows:

  \[ 0 = \dot{f}^a_{\alpha} + f^b_{\alpha} \dot{x}^c F^a_{bc} + (\dot{x}^d N^c d + \dot{f}^c_0) C^a_{bc} = \nabla (\dot{x}, \dot{f}^0) f^a_{\alpha}. \quad (10) \]

  \( \Rightarrow \) Frame \( f \) is parallely transported.
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\[ \Rightarrow \text{Frame } f \text{ is parallely transported.} \]
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• Symmetry breaking: $(y^a) = (y^0, y^1, y^2, y^3) = (0, 0, 0, 0)$, $y^4 = 1$. 

• Cartan curvature $F^{ab}$ gives Palatini action.

• Idea: symmetry breaking field $y^A = \bar{\Psi} \Gamma^A \Psi$ with higher spinor $\Psi$. 

Stelle-West action uses $\mathfrak{g}$-valued curvature form $F^{AB}$:

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Dirac-Cartan-Higgs action and symmetry breaking

- **Ingredients:**
  - Spin groups $\tilde{G}$ and $\tilde{H}$: double covers of $G$ and $H$.
  - Spin frame bundle $\tilde{P}$ is principal $\tilde{H}$ bundle.
  - Right action of $\tilde{H}$ on $\tilde{G}$ yields principal $\tilde{G}$-bundle $\tilde{Q} = \tilde{P} \times_{\tilde{H}} \tilde{G}$.
  - Levi-Civita connection gives Ehresmann connection $\tilde{A}^{AB}$ on $\tilde{Q}$.
  - “Higher” spinor $\Psi$: section of bundle associated to $\tilde{Q}$.
  - Dirac matrices $\Gamma^A$ acting on $\Psi$.

- **Construction of Dirac-Cartan-Higgs action:**
  - Exterior covariant derivative of higher spinor:
    \[
    D\Psi = d\Psi + \frac{1}{8} \tilde{A}^{AB} \wedge [\Gamma^A, \Gamma^B] \Psi.
    \] (13)
  - Dirac-Cartan operator:
    \[
    \mathcal{D}\Psi = \rho([E^A, E^B]) \tilde{A}^{AB} \mathcal{D}\Psi.
    \] (14)
  - Dirac-Cartan-Higgs action:
    \[
    L = \bar{\Psi} \mathcal{D}\Psi + m \bar{\Psi}\Psi - \lambda (\bar{\Psi}\Psi)^2.
    \] (15)
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  - Dirac-Cartan operator:
    \[ \slashed{D} \Psi = \rho([\slashed{E}^A, \slashed{E}^B]) \tilde{A}^{AB} - D\Psi. \]  \hspace{1cm} (14)
  - Dirac-Cartan-Higgs action:
    \[ \mathcal{L} = \bar{\Psi} \slashed{D} \Psi + m\bar{\Psi} \Psi - \lambda (\bar{\Psi} \Psi)^2. \]  \hspace{1cm} (15)
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• Lorentz group $H$ contains rotation subgroup $K \subset H$.
• Principal bundle $P$ of frames reduces to $K$-bundle.
• Base space: “observer space” $O$ of normalized velocities.
• Transition to spin groups $\bar{G}$, $\bar{H}$, $\bar{K}$.
• Cartan connection contains $\mathfrak{k}$-valued part.

$\Rightarrow$ Ashtekar variables.